Convolution of Scale Invariant Continuous Ranked Probability Scores for Testing Experts' Statistical Accuracy

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Abstract

Computable solutions for expectations of Continuous Ranked Probability Scores are presented. After deriving a scale invariant version of these scores, a closed form for the convolutions of scores is presented. This closed form enables the testing experts' statistical accuracy. Results are compared with tests using a familiar Chi-square goodness of fit test using a recent data set of 6,761 expert probabilistic forecasts for which true values are known.

Keywords expert judgment, scoring rules, continuous ranked probability score, probability interval score, geometric probability, classical model, overconfidence, location bias.

1 Introduction

Scoring rules were introduced by de Finetti in 1937 as tools for encouraging honesty in eliciting subjective probabilities (De Finetti, 1937) and have been further developed by many authors including Shuford et al. (1966); Savage (1971); Murphy (1977); Brown (1974); DeGroot and Fienberg (1983); Gneiting and Raftery (2007). The latter reference gives an extensive overview. An expert receives a score as a function of his/her probability assessment and the realization. The score is strictly proper if the expert maximizes (for negatively sensed rules, minimizes) his/her expected score per item by, and only by, stating his/her true belief. Using a result of Murphy (1977), DeGroot and Fienberg (1983) gave an additive decomposition of strictly proper rules into 'calibration' and 'refinement' terms, thereby replacing Murphy's 'resolution' (refinement applies only to well-calibrated experts). In the case of the logarithmic rule, refinement becomes Kullback-Leibler divergence of the sample distribution for an unknown quantity and forecasters attempt to predict this distribution. Scoring rules for sets of variables play a key role in the classical model (CM, Cooke (1991)) for combining expert judgments.

Scoring rules for individual variables were not designed for evaluating or combining experts and are not generally fit for that purpose. Indeed, rewarding honesty is not the same as rewarding quality. A simple example illustrates this difference: Consider 100 fair coin tosses. An expert assesses the probability of *heads* on each toss as 1/2. With the standard scoring rules, the score for the outcome *heads* is the same as their score for *tails* on each toss. If the score for all 100 assessments is a function of their 100 scores for the individual tosses, then their score for 100 tosses is independent of the outcome sequence; the outcome of 100 *heads* receives the same score as 50 *heads* and 50 *tails*. Equal scores do not imply equal quality.

Another example concerns the quadratic rule for 'rain / no rain' events. This rule is positively sensed on [-1, 1] and assigns the quadratic score $2r-r^2-(1-r)^2$ if rain occurs, where r is the expert's probability of rain. Interchange r and (1-r) in case it does not rain. Consider 1000 next day forecasts of rain by two experts. Suppose the experts bin their forecasts as shown below (Cooke, 2014).

Probability of rain next day		5%	15%	25%	35%	45%	55%	65%	75%	85%	95%	Totals
expert 1	assessed	100	100	100	100	100	100	100	100	100	100	1000
	realized	5	15	25	35	45	55	65	75	85	95	500
expert 2	assessed	100	100	100	100	100	100	100	100	100	100	1000
	realized	0	0	0	0	0	100	100	100	100	100	500

Table 1: Binned subjective probability of rain assessments for the 1000 next days by two experts.

Ten probability bins are considered, each associated with a forecast probability of rain. The experts' assessments are equally informative in the sense that they assign the same probabilities to the same number of days. Expert 1 is statistically perfectly accurate, that is, the empirical frequency of actual rainy days from the assessed 100 days is identical with the probability associated with each bin. Expert 2 is massively inaccurate statistically. The sample distributions bear little resemblance to his/her assessed probabilities $(5\%, \ldots, 95\%)$. Expert 1 has an average quadratic score of 0.67 and Expert 2 an average quadratic score of 0.84. Expert 2 gets a better quadratic score because the higher resolution in the sample out-weighs statistical inaccuracy. Such examples make it difficult to explain intuitively what the scores 0.67 and 0.84 mean. For more discussion, see (Cooke, 1991, 2014). In the context of expert judgment we would like to reward both honesty and quality with scoring rules that are intuitive and easily explained. This requires numerical insight into the rules' behavior.

The negatively sensed Probability Interval Score (PIS) and its related Continuous Ranked Probability Scores (CRPS) have recently been applied to COVID-19 probabilistic predictions (Ray et al., 2020). Gneiting and Raftery (2007) note: "Applications of the *CRPS* have been hampered by a lack of readily computable solutions to the integral (1)" (see below).

This article presents computable solutions to this integral which then allow us to study its trade offs between statistical accuracy and "sharpness". The *CRPS* can be thrown into a scale invariant form which offers significant advantages. After reviewing the *PIS* and *CRPS*, we introduce a re-parametrization of *CRPS* by transforming the realizations according to the probability integral transformation of an expert's assessed cumulative distribution function (*CDF*). An expert with *CDF* F_X for continuous variable X is scored not with respect to the realization y but with $F_X(y)$, the quantile of the distribution of X realized by y. The proposed CRPS transformation has several advantages:

- (i) The transformed *CRPS* becomes scale invariant.
- (ii) The expert's sampling distribution of transformed CRPS can be expressed in closed form
- (iii) The density of convolutions of transformed *CRPS* scores for independent variables is available in closed form.
- (iv) Transformed CRPS can then be used to test the expert's statistical accuracy without recourse to an asymptotic distribution.

On the downside, CRPS is insensitive to location bias. If the experts assess only certain quantiles, a second downside is that continuous CDFs must be interpolated before applying CDFs.

After introducing the Probability Interval Score and the Continuous Ranked Probability Score, computable examples of the latter are given. This motivates a scale invariant version of CRPS to be used in testing experts' "statistical accuracy", a term denoting goodness-of-fit tests adapted to expert judgments. The closed form of the convolution of scale invariant CRPS scores is introduced. Using a recently compiled database of expert judgments with realizations (Cooke et al., 2021), the results of this test are compared with the statistical accuracy score of CM. The scores are also compared with regard to rewarding proximity of the medians (considered as point forecasts) to the realizations. We conclude that the scale invariant CRPS better rewards proximity to the median while failing to punish location bias. The closed form convolution also confers advantages with respect to CM.

2 Probability Interval Scores

Numerical insight into the behavior of these scores requires a bit of effort. For the $(1 - \alpha)$ uncertainty interval [L, U], with upper (lower) bound U(L), the *PIS* (negatively sensed) (Aitchison and Dunsmore, 1968) for realization y is

$$(U-L) + \frac{2}{\alpha} \times [(L-y)_+ + (y-U)_+]$$

where $X_+ = X$ if X > 0 and $X_+ = 0$ otherwise. Note that $s = 2/\alpha$ is the slope of the overconfidence penalty for $y \notin [L, U]$. The length ||U - L|| is called the "sharpness"; small values reward concentrated probability mass.

To better understand the characteristics of *PIS*, consider $Y \sim U[0, 1]$ and the $(1-\alpha)$ uncertainty interval [L, U]. Then

$$E_Y[PIS(y)] = U - L + \frac{2}{\alpha} \int_0^L (L - x) \, \mathrm{d}x + \frac{2}{\alpha} \int_U^1 (x - U) \, \mathrm{d}x = U - L + \frac{1}{\alpha} \left[L^2 + (U - 1)^2 \right].$$

For the central 0.9 interval [0.05, 0.95], the expected PIS is 0.95. Suppose an expert prefers to give an 80% interval, [0.1, 0.9], then the expected score is 0.9. This is better than 0.95 because the prediction interval is sharper. An expert seeking to optimize (i.e., minimize) his/her expected score might take a central 2% prediction interval [0.49, 0.51] with expected score of 0.51. The way in which the *PIS* trades overconfidence for sharpness may strike some as counter-intuitive. For example, an expert claiming that the degenerate interval [0.5, 0.5] has 40% probability of catching the realization would achieve an expected score of 0.833, better than the score of the 90% central interval. The sharpness of an interval of zero length outweighs the overconfidence of claiming 40% mass at the point 0.5.

3 Continuous Ranked Probability Score

Consider y an unknown scalar quantity of interest. Suppose y has a true forecast cumulative distribution function (CDF) F_Y , characterizing the distribution of a random variable Y, which is not known. An expert provides CDF F_X which (s)he believes to be the distribution of Y. We assume both F_Y and F_X are continuous and strictly increasing on their support. The continuous ranked probability score (CRPS) is defined as (Brown, 1974)

$$CRPS(F_X, y) = \int_{-\infty}^{\infty} [F_X(x) - 1_{\{x \ge y\}}]^2 dx.$$
(1)

Lower values indicate better performance. CRPS is known to be strictly proper relative to a class of Borel probability measures with finite first moment (Gneiting and Raftery, 2007). As mentioned in the introduction, lack of readily computable solutions for (1) have restricted the use of CRPS score. To understand the behavior of the CRPS score, let us consider $X \sim U[L, H]$, with 0 < L < H < 1. As we will show later, this particular choice of distribution is relevant for the development of our proposed score. For $y \in [0, 1]$:

$$CRPS(F_X, y) = \begin{cases} L - y + \frac{H - L}{3} & for \ 0 \le y < L \\ \frac{(y - L)^3}{3(H - L)^2} - \frac{(y - H)^3}{3(H - L)^2} & for \ y \in [L, H] \\ y - H + \frac{H - L}{3} & for \ H < y \le 1. \end{cases}$$
(2)

Figure 1 shows the CRPS score as a function of y, for different cases of L and H. Cases when yfalls within and outside the F_X support are highlighted.



Figure 1: CRPS score for $X \sim U[L, H]$ and $y \in [0, 1]$. Differences for y < L (red), for $y \in [0, 1]$ [L, U](blue), and y > U(green) are highlighted.

The expectation of the CRPS score, which may be infinite, is given by

$$E_Y[CRPS(F_X, Y)] = \int_y \int_x [F_X(x) - 1_{\{x \ge y\}}]^2 \mathrm{d}x \,\mathrm{d}F_Y(y).$$
(3)

We discuss some computable solutions for this expectation.

3.1Computable solutions

Consider $Y \sim U[0,1]$ and an assessment of Y's distribution by an expert as that of random variable X, $X \sim U[0, H], 0 < H < 1$. The expert thinks values greater than H are impossible, although these can in fact arise. The expected CRPS is computed based on the distribution of Y. The CDFof X, F(x) = x/H, for $x \in [0, H]$ and $F_X(x) = 1$, for $x \ge H$, along with the survivor function of X, S(x) = 1 - F(x) are shown in Figure 2 (see also Candille and Talagrand (2005)). Then $E_Y[CRPS(F_X, Y)] = \int_0^1 \int_0^1 [F(x) - 1_{x \ge y}]^2 dx dy$ is computed in 2 steps:



Figure 2: Cumulative distribution and survivor function of an uniformly distributed random variable on [0,H].

A) For y < H:

$$\int_{0}^{H} \left[\int_{0}^{y} \frac{x^{2}}{H^{2}} dx + \int_{y}^{H} \left(\frac{H-x}{H} \right)^{2} dx \right] dy = \int_{0}^{H} \left[\frac{y^{3}}{3H^{2}} + \int_{H-y}^{0} \frac{z^{2}}{H^{2}} (-dz) \right] dy = \frac{H^{2}}{6}$$

B) For y > H:

$$\begin{split} \int_{H}^{1} \left(\int_{0}^{H} \frac{x^{2}}{H^{2}} dx + \int_{H}^{y} dx + \int_{y}^{1} 0 \, dx \right) dy &= \int_{H}^{1} \left(\frac{H}{3} + y - H \right) \, dy = \frac{H(1 - H)}{3} + \int_{0}^{1 - H} z dz \\ &= \frac{H(1 - H)}{3} + \frac{(1 - H)^{2}}{2}. \end{split}$$

Therefore:

$$E_Y[CRPS(F_X, Yy)] = \frac{H^2}{6} + \frac{H(1-H)}{3} + \frac{(1-H)^2}{2}.$$
(4)

As noted by Hersbach (2000), these results acquire a physical dimension. The result for (A) is the score an expert with $X \sim U[0, H]$ expects, namely $H^2/6$, which has the physical dimension of H^2 . If X is in meters and changes to centimeters, the expected score increases by a factor 10^4 .

If $X \sim U[L, H]$, for 0 < L < H < 1, then the same method of calculation applies mutatis mutandis. If L = 1 - H, with $H \ge 0.5$, then the contributions from x < y and y < x are equal and we need only to double the contribution from x < y. If y < L, the contribution from x < y is zero. We compute

$$\begin{split} \int_{L}^{H} \int_{L}^{y} F(x)^{2} \, dx \, dy + \int_{H}^{1} \int_{L}^{y} F(x)^{2} \, dx \, dy = \\ \int_{L}^{H} \int_{L}^{y} \frac{(x-L)^{2}}{(H-L)^{2}} \, dx \, dy + \int_{H}^{1} \left[\int_{L}^{H} \frac{(x-L)^{2}}{(H-L)^{2}} \, dx + \int_{H}^{y} \, dx \right] \, dy = \\ \int_{L}^{H} \frac{y^{3}}{3(H-L)^{2}} \, dx + \int_{H}^{1} \left[\frac{(H-L)}{3} + (y-H) \right] \, dy = \\ \frac{(H-L)^{2}}{12} + \frac{(H-L)(1-H)}{3} + \frac{(1-H)^{2}}{2}. \end{split}$$

Adding the identical contribution from y < x gives:

$$E_Y[CRPS(F,Y)] = \frac{(H-L)^2}{6} + \frac{2(H-L)(1-H)}{3} + (1-H)^2.$$
(5)

Again, the score inherits a physical dimension from X. Substituting L = 1 - H in (5) we find that $E_Y[CRPS(F,Y)]$ for $X \sim U[1-H,H]$ is equal to $E_Y[CRPS(F,Y)]$ for $X \sim U[0,H]$ from (4). This holds for any 0 < L < H < 1, with L = 1 - H. Hence, for $X \sim U[0,0.7]$, $E_Y[CRPS(F,Y)] =$ $0.1966 = E_Y[CRPS(\tilde{F},Y)]$, for $\tilde{X} \sim U[0.3,0.7]$. By the same token, $X \sim U[0,0.5]$ yields the same expected score of 1/4 as \tilde{X} with distribution $\delta(0.5)$ concentrated at 0.5.

If $F_i \to F$, then $E_Y[CRPS(F_i, y)] \to E_Y[CRPS(F, y)]$, by the Helly-Bray theorem (Billingsley, 2013). It follows that, if $F_n \to U[0, 0.5]$ and $\tilde{F}_n \to \delta(0.5)$, then for all $\varepsilon > 0$ and for all sufficiently large n, $|E_Y[CPRS(F_n, Y)] - E_Y[CPRS(\tilde{F}_n, Y)]| < \varepsilon$. This illustrates how the CRPS compensates loss of statistical accuracy by a gain in "sharpness", and again illustrates that equal scores do not entail equal quality.

Note that if the probabilistic forecast is the distribution of Y (uniform [0,1]), then the expected CRPS score is 1/6, by (4). For 0 < H < 1, $E_Y[CRPS(F,Y)] > \frac{1}{6}$, by (4) and for 0.5 < H < 1, L = 1 - H, $E_Y[CRPS(F,Y)] > \frac{1}{6}$, by (5). An expert would receive a better (lower) expected score if their probabilistic forecast were equal to the distribution of Y.

4 Test for statistical accuracy

We would like to test the hypothesis that Y follows the expert's assessed distribution F_X . Applying the probability integral transformation, let $U = F_X(X)$ and define $V = F_X(Y)$. Then $F_U(u) = u$. The hypothesis that $F_X = F_Y$ is equivalent to the hypothesis

$$H_0: V \sim U[0, 1].$$

In this case CRPS can be written, for the realization v and for U uniformly distributed on [0, 1], as

$$CRPS(F_U, v) = \int_{-\infty}^{\infty} [u - 1_{\{u \ge v\}}]^2 du$$

= $\int_{-\infty}^{0} (0 - 0)^2 du + \int_{0}^{v} u^2 du + \int_{v}^{1} (u - 1)^2 du + \int_{1}^{\infty} (1 - 1)^2 du$
= $\frac{v^3}{3} - \frac{(v - 1)^3}{3}.$ (6)

The range of the $CRPS(F_U, v)$ is $\left[\frac{1}{12}, \frac{1}{3}\right]$, for $v \in [0, 1]$ and $U \sim U[0, 1]$. The distribution of CRPS is the distribution of the random variable

$$\frac{1}{3} \left[V^3 - (V-1)^3 \right] = \frac{1}{3} - V + V^2,$$

taking values in $[\frac{1}{12}, \frac{1}{3}]$ (lower values are better). Under the null hypothesis, V is uniform [0, 1]. For fixed $Q \in [\frac{1}{12}, \frac{1}{3}]$, to find the probability that $CRPS(F_U, v) \leq Q$, under the null hypothesis, we find the roots of $V^2 - V + (\frac{1}{3} - Q) = 0$:

$$v_{1,2} = \frac{1 \pm \sqrt{1 - 4(\frac{1}{3} - Q)}}{2} = \frac{1 \pm \sqrt{4Q - \frac{1}{3}}}{2}$$

Collecting the mass between the two roots, we obtain the CDF

$$P(CRPS(F_U, v) \le x) = \sqrt{4x - \frac{1}{3}},$$

with density

$$\frac{2}{\sqrt{4x - \frac{1}{3}}}, \ x \in \left(\frac{1}{12}, \frac{1}{3}\right).$$
(7)

Figure 3 shows $CRPS(F_U, v)$, for $v \in [0, 1]$ and $U \sim U[0, 1]$, together with its CDF and density under the null hypothesis H_0 .



Figure 3: CRPS function, for $U \sim U[0,1]$ and $v \in [0,1]$, together with its cumulative and density functions, under the null hypothesis H_0 .

From Figure 3 it is evident that the score $CRPS(F_U, v)$ is symmetric around the value v = 0.5. This is different from the behavior of $CRPS(F_U, v)$ exhibited in Figure 1a or 1b, and it illustrates a feature of the scale invariant version of CRSP.

From equation (7), we can easily compute

$$E_{V} [CRPS(F_{U}, V)] = \frac{1}{6}$$

$$E_{V} [CRPS^{2}(F_{U}, V)] = \frac{1}{16} \left(\frac{1}{5} + \frac{1}{3}\right)$$

$$Var_{V} (CRPS(F_{U}, V)) = 0.0055555$$

It is handier to consider the following transformation

$$Z(U,V) = 4CRPS(F_U,V) - \frac{1}{3}.$$
(8)

Then Z has CDF and density

$$F_Z(x) = \sqrt{x}$$
, and $f_Z(x) = \frac{1}{2\sqrt{x}}, x \in [0, 1].$

Note that f_Z is the density of U^2 , where $U \sim U[0, 1]$. So far, only one unknown scalar quantity of interest, and expert's resulting *CRPS* score, have been considered.

Suppose an expert provides uncertainty assessments for n random variables. The emerging question is how to aggregate the *CRPS* scores of each of the n variables? The transformation (8), and the observation that the density f_Z is the density of a squared uniform random variable are again handy.

If we assume the *n* variables to be independent, then we need to consider Z_1, \ldots, Z_n independent variables, each with density f_Z . For these, we need to find the density of $Z^{(n)} = Z_1 + \cdots + Z_n$. Or, in terms of the squared uniform random variables, we need to find the density of $S_n = U_1^2 + \cdots + U_n^2$.

Weissman (2017) provides closed form distributions for S_n , for n = 3, 4, 5, 6, 8, 10, 12 and their graphical representations. A connection is also made with a topic of geometrical probability, that is, finding the cumulative distribution function of S_n , $P(S_n \leq s)$ is equivalent to finding the volume of the intersection between the unit n-cube and the ball of radius \sqrt{s} , in \mathbb{R}^n , when both are centered at the origin. In his comment to Weissman (2017), Forrester (2018) observes that the more generic volume problem posed by Xu (1996), of finding the volume of the intersection of a cube and a ball in n-space has already been solved by B. Tibken and D. Constales (Rousseau and Ruehr (1997)). Weissman (2017) reports that Constales' solution to the volume problem involves a method based on Fourier series and implies that, for general n,

$$F_n(s) = P(S_n \le s) = \frac{1}{6} + \frac{s}{n} + \frac{1}{\pi} \mathbf{Im} \sum_{k=1}^{\infty} \left[\left(\frac{C(2\sqrt{k/n}) - iS(2\sqrt{k/n})}{2\sqrt{k/n}} \right)^n \frac{e^{2\pi i k s/n}}{k} \right], \quad (9)$$

where $S(x) = \int_0^x \sin(t^2) dt$ and $C(x) = \int_0^x \cos(t^2) dt$ denote the Fresnel integrals and **Im** is the imaginary part of a complex number.



Figure 4: The exact distribution (red) and empirical distribution function of the sum of squared simulated uniform observations (blue), for n=2 (left) and n=10 (right).

Figure 4 graphically compares the above cumulative distribution for n=2 (left) and n=10 (right) with the empirical distribution function of the corresponding sum of squared uniform observations. 100 observations were sampled for both empirical distribution functions. The cumulative distribution function was implemented in **R**, by making use of functions implementing the Fresnel integrals in the **pracma** package (Borchers and Borchers, 2022).

5 Expert data

In this paper, we use data from 49 studies involving 526 experts assessing in total 580 calibration variables from their fields for which realizations are known (four experts from the original data were dropped because they did not assess all calibration variables in their respective panels). In total there are 6,761 expert probabilistic forecasts of variables from their fields for which true values are known. The data is described and referenced in (Cooke et al., 2021). The supplementary information for that reference gives a description of the Classical Model, whose relevant aspects are briefly reviewed here.



Figure 5: Number of experts per number of assessed calibration variables.

The number of assessed calibration variables differ per study and Figure(5) provides information about this. Experts assessed at least 7 and at most 21 calibration variables during all studies, and 10 calibration were used in 21 of the 49 studies.

6 Results of Statistical Tests

This section compares statistical accuracy (SA) tests based on the classical model (CM, Cooke (1991)) versus based on CPRS. In the classical model, SA is measured as the probability of falsely rejecting the hypothesis that a probabilistic assessor is statistically accurate. It is, in other words, the P-value of rejection for this hypothesis. We hasten to add that CM does not test and reject expert hypotheses but, in compliance with proper scoring rule theory for sets of assessments, uses this P-value to measure the degree of correspondence between assessments and data in forming weighted combinations of expert distributions.

In the data used for this analysis, expert assessments take the form of fixed quantiles, 5%, 50%,, 95%, from the assessor's subjective distribution for a continuous unknown quantity. When n true values for a number of such quantities are observed, we compute the sample distribution s of interquantile relative frequencies and compare this with the theoretical inter-quantile mass function p = (0.05, 0.45, 0.45, 0.05). The test statistic is 2nI(s, p), where I is the Shannon relative information (log likelihood ratio) and n is the number of calibration variables. Assuming that the realizations are independently sampled from the assessor's distributions, this statistic is asymptotically chi-square distributed with degrees of freedom equal the number of assessed quantiles. Thus, SA is measured as $1 - F_{\chi^2}(2nI(s, p))$ where F_{χ^2} is the CDF of a χ^2 distribution with three degrees of freedom. High scores (near 1) are good, low scores (near 0) mean it is unlikely that the divergence between s and p should arise by chance. Note that CM relies on an asymptotic approximation which for a small number of observations is not very good (Cooke, 2014). Simulations for ten calibration variables are provided in (Hanea and Nane, 2021). It is deemed capable of detecting only large differences in experts' performances. It uses only the assessed quantiles and does not rely on an interpolated CDF.

When applied to this expert judgment data, a test based on CRPS must interpolate an expert's CDF. For this purpose we follow CM and adopt a minimally informative distribution relative to a uniform support (chosen by the analyst¹) which complies with the expert's quantile specification. The resulting CDF can be found in Hanea and Nane (2021). CM uses this interpolation for computing an expert's informativeness, but not for computing SA.

Consider n observations of continuous variables assessed by an expert e. The following procedure calculates SA for the CRPS statistic

- 1. For each realization $y_i = 1, ..., n$, compute $q_e^i = F_{i,e}(y_i)$, the quantile of y_i in expert e's CDF $F_{i,e}$.
- 2. The SA hypothesis H_0 entails that these quantiles are independent samples from a uniform distribution. Under this hypothesis, $CRPS_i(e) = CRPS(F_U, q_e^i)$ can be computed from (6).
- 3. For each i = 1, ..., n, compute $z_i(e) = 4CRPS_i(e) 1/3$, which is given in (8)
- 4. Compute $s(e) = F_n(\sum_{i=1}^n z_i(e))$, where F_n is the exact distribution of the sum of *n* independent squared uniform variables, given in (9).

Note that the procedure can be applied for continuous and invertible CDFs. CRPS uses an exact instead of an asymptotic distribution for the convolution of these CDFs. From Figure 3 it is evident that the score CRPS(v) for value v is symmetric around the value 0.5. The distribution of the sum of such variables is insensitive to location bias in the following sense: the score for 2n observations of 0.4 is the same as for n observations of 0.4 and n observations of 0.6. Figure 3 also shows that CRPS is also insensitive to under-confidence: An expert whose probability transformed realizations are all 0.5 scores better than one for whom the hypothesis of Section 4 that $F_X = F_Y$ holds. Under-confidence is rare with expert judgment. Over-confidence, on the other hand, is not rare and the CRPS score is sensitive to over-confidence (see Figure 6).

Figure 7 plots the SA scores for 526 experts based on CM and on CRPS. Although the drift of the two scores is similar, there is substantial scatter. CRPS's log geomean SA score is -2.76 while that of CM is -3.47; in this sense CRPS is less severe.

We define an expert's location bias as the absolute difference between the percent of realizations above the medians and 50%. Location bias of 50% means that all realizations are above or all realizations are below the medians. Figure 8 black circles the CM scores of those experts for whom the location bias is greater or equal to 20%. CM SA scores of experts for whom the location bias is 50% (all realizations were either below or all above the medians) are red circled. The location bias of these circled experts is missed by CRPS and helps explain some of the downward scatter.

For 70 experts, the location bias is 0%. These are termed experts without location bias (though of course there could be location bias in the lowest and highest inter-quantile intervals). Figure 9

 $^{^{1}}$ In principle, any background measure supporting the experts' quantiles and realization may be used, this data imposes the uniform background for convenience



Figure 6: Frequency of percentiles of realizations for 6761 expert probabilistic forecasts from 49 studies.



Figure 7: Statistical accuracy of 526 experts with respect to CRPS (red) and CM (blue). SA scores are ordered by CRPS SA scores.

plots these 70 CM SA scores against all the CRPS SA scores. On this subset, CRPS's log geomean SA score is -2.50 while that of CM is -2.68.

The number of calibration variables assessed in the 49 panels ranges from 7 to 21, see Figure 5. This influences CM SA in two ways, a larger number (i) increases the accuracy of the χ^2 approximation and (ii) tends to lower SA scores of poorly calibrated experts as the test of statistical accuracy has greater power. The first should decrease the differences between CM SA and CRPS SA, at least for location unbiased experts, whereas the second enables a greater range of CM SA scores and might therefore increase the differences. A multiple regression of $\log(\frac{CRPSSA}{CMSA})$ against location bias and number of calibration variables explains 30% of the variance (adjusted R^2) and both explanatory variables have a significant positive effect on the dependent variable. Thus, the influence



Figure 8: Statistical accuracy of 526 experts with CM and CRPS with location biases circled. Location bias is the absolute difference between the percentage of medians above the realizations and 50%. Black circles denote CM scores of those experts for whom the location bias is greater or equal to 20%. Red circles denote location bias of 50%.



Figure 9: Statistical accuracy of 526 experts with CRPS and 70 experts with CM without location bias.

of (ii) exceeds that of (i). The Pearson correlations of the dependent variable with location bias and with number of calibration variables are 0.41 and 0.29 respectively. The correlation of the two explanatory variables is -0.17. A more detailed analysis might better explain the differences in the two SA scores but at this point it appears that location bias is the major factor.

7 MAPE

Statistical accuracy is not the only scoring variable of interest; the proximity of the median to the realizations is also important. There are many measures for such proximity (Morley et al., 2018; Gneiting et al., 2007), each with benefits and drawbacks. Perhaps the most popular is the Mean Absolute Percentage Error (MAPE), defined for forecasts x_i and realization r_i i = 1...n as

$$\frac{1}{n}\sum_{i=1}^{n}|\frac{x_i-r_i}{r_i}|.$$

This is evidently unstable for very small r_i . Instability arises on this data set, as the largest MAPE is over one million. The MAPEs of 326 experts were less than 2 and we focus on this subset. Figure 10 plots these MAPE scores (left axis) and also plots the corresponding values of CMSA and CRSPSA (right axis). Although not overwhelmingly clear in the figure, the CRSPSA scores tend to be higher than those of CMSA, especially for very low MAPEs (see trend lines). The Spearman correlation of CMSA and MAPE on this data subset is -0.15 while that of CRSPSA and MAPE is -0.26. This results from the fact that CRPS uses the (interpolated) CDF whereas CM is based on inter-quantile hit-rates. It is reasonable to expect that weighing experts according to CRPS scores will produce better MAPE values for the combination of experts than CM. Other researchers (Flandoli et al., 2011) have used likelihood scores based on interpolated CDFs and achieved better MAPE performance than with CM, but such scores are notoriously improper. The great advantage of CRPS in this regard is that it is based on a strictly proper scoring rule.



Figure 10: 326 MAPE scores < 2 (left axis) and SA for CM and CRPS (right axis), with trend lines.

8 Conclusion

A scale invariant version of the Continuous Ranked Probability Score (CRPS) has been developed and applied to an expert judgment data base involving 49 studies with 526 experts assessing in total 580 calibration variables from their fields for which realizations are known. The transformed CRPSc yields a test for experts' statistical accuracy which has the advantage of a closed form solution without appeal to an asymptotic distribution. Compared to the statistical accuracy test used in the classical model it has the advantage of better rewarding proximity of a median point forecast to the realization. Nonetheless, the CRPS is insensitive to location and under-confidence bias. A future study will focus on combinations of experts' judgments, comparing the performance of CRPS with other tests based on the Chi Square, the Kolmogorov Smirnov and the Cramer Von Mises statistics.

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