Crib Sheet David Randall Atmosphere, Clouds and Climate Princeton U Press 2012.
Cooke 10-17-2017

Units

K = Kelvin; W = J/s: n_i = # mol; N_i = # particles, T temp[K],
Joule = [Newton × m] = (kg m s^{-2}) × m = [kg m^2/s^2]
Pa = 1 Newton/m^2 = [F/m^2] = [kg m s^{-2}/m^2] = [kg/(ms^2)].
1 atmosphere = 101.325 kPa = average air pressure at 45^{o}N
$N_i k/n_i := R^*$ = ideal gas constant.
Boltzmann constant k = 1.38064852 × 10 $^{-23}$ [J/K],
$R^* = 8.3144598$ J·mol^{-1}·K^{-1}
σ_{SB} = Stefan-Boltzmann constant = 5.67×10^{-8}W$m^{-2}K^{-4}$
Avogadro's Nr = $R^*/k = N_i/n_i = 6.022\times10^{23}$/mol.

Partial Pressure of gas i p_i : $p_iV = N_ikT$;

$p_i = N_ikT/V = n_iR^*T/V$; ρ_i = density = M_i/V; M_i = mass of i = n_im_i; m_i = molecular mass.

$p_i = \rho_in_iR^* T/M_i = \rho_iR^* T/m_i$.

Hydrostatic law: z = height

$\delta p_i/\delta z = -\rho_ig = p_im_ig/(R^*T)$; $D[\ln(p_i)] = \partial p_i / p_i = m_ig/(R^*T)$. if T doesn't depend on z:

$p_i = \exp(-zgm_i/R^*T) = e^{-z/H_i}$)' $H_i = R^*T/m_i$ = characteristic height.

Radiation Budget [Wm^{-2}]: (from Trenberth, IPCC)

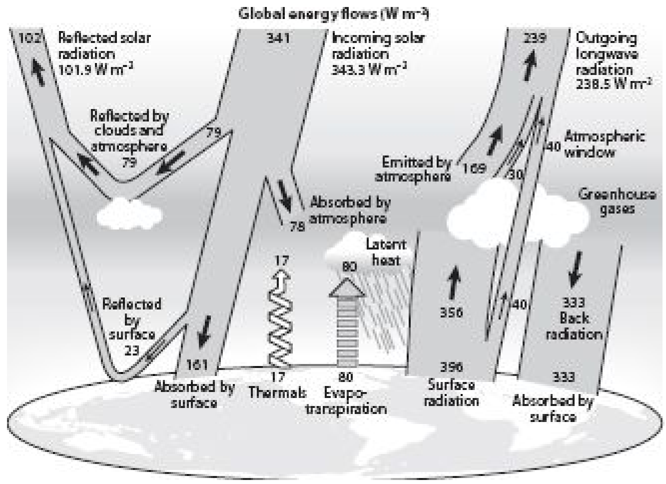


Figure 1.3. An overview of the flow of energy in the climate system.

The global annual mean Earth's energy budget for the March 2000 to May 2004 period ($W\ m^{-2}$). The broad arrows indicate the schematic flow of energy in proportion to their importance.

Source: Based on a figure in Trenberth et al. (2009).

$S = 1365 Wm^{-2}$.= incoming solar energy per unit area per unit time on circular cross section. $S*\pi a^2$ is *incident energy*; $a = 6371$km radius of Earth. Divide by area of Earth's surface $= 4\pi a^2 =$ $S/4 = 341.25$. Randall uses 343 Wm^{-2}.

Incoming: Wm^{-2}

343 – 102	=	241. 241 – 78	=	163
Albedo= =79 + 23 Clouds+surface		absorbed by clouds		absorbed by surface

Into clouds

78	+	17	+	80	+	356	=	531
From Sun		thermals		evaporation		from surface		

Clouds to space

531	–	333	=	198
		Back to earth		

Net out

198	+	40	=	238.5
From clouds		through window		net out (rounding error restored)

Planetary energy balance

Net flow of radiation across top of atmosphere N, a = albedo, ε_{Bulk} = bulk emissivity:

$N = \frac{1}{4}S(1\text{-}a) - \varepsilon_{Bulk}\,\sigma_{SB}\,T_{surf}^4$, σ_{SB} is the Stefan-Boltzmann constant = $5.67\times10^{-8}Wm^{-2}K^{-4}$.

The value of the Stefan–Boltzmann constant is derivable as well as experimentally determinable; see Stefan–Boltzmann law for details. It can be defined in terms of the Boltzmann constant as:

$$\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2} = \frac{\pi^2 k_B^4}{60\hbar^3c^2} = 5.670373(21)\;\cdot 10^{-8}\;\mathrm{J\,m^{-2}\,s^{-1}\,K^{-4}}$$

where:

- k_B is the Boltzmann constant;
- h is the Planck constant;
- $\hbar$ is the reduced Planck constant;
- c is the speed of light in vacuum.

In equilibrium N = 0 (actually we think – $1Wm^{-2}$, though this is within measurement error)
In Equilibrium (drop subscripts)

$\frac{1}{4}S(1\text{-}a) = \varepsilon\sigma T^4$; $S = 4\varepsilon\sigma T^4/(1\text{-}a)$

Change with small perturbation

$\Delta N = \frac{1}{4}\Delta S(1\text{-}a) - \frac{1}{4}S\Delta a - 4\varepsilon\sigma T^4\Delta T/T - \Delta\varepsilon\sigma T^4 = 0$ in new equilibrium:

$4\varepsilon\sigma T^4\Delta T/T = \frac{1}{4}\Delta S(1\text{-}a) - \frac{1}{4}S\Delta a - \Delta\varepsilon\sigma T^4$ [use $\frac{1}{4}S(1\text{-}a) = \varepsilon\sigma T^{4]}$]

$= \varepsilon\sigma T^4\,(\Delta S/S) - \Delta a\varepsilon\sigma T^4/(1\text{-}a) - \Delta\varepsilon\sigma T^4$, or

$\Delta T/T = \frac{1}{4}[(\Delta S/S) - \Delta a/(1\text{-}a) - \Delta\varepsilon/\varepsilon]$.

Suppose $\Delta a = \Delta\varepsilon = 0$. T = 288K, S = $240Wm^{-2}$. Solar constant varies by 0.1%:
$\Delta T = 288(.24/240)\frac{1}{4} = 0.017C$. Not much effect on Temperature.

Suppose $\Delta S = \Delta a = 0$. Doubling CO2 $\Longrightarrow$ reduce OLR by $4Wm^{-2}$:
$\Delta T = 288(4/240)\frac{1}{4} = 1.22C$. (neglects all feedbacks)

[If we take the earth as a black body radiating at 255k, no atmosphere, $N = R[Wm^{-2}] = \sigma T^4$; $\sigma = 5.67\times10^{-8}\,[Wm^{-2}/K^4]$; $\lambda(T) = dT/dR = (4\sigma T^3)^{-1}$.

without feedback: $T = 255$, $R = 239.7418$; *put* $\Delta R = 4$,
$\Delta T_o = ((R+4)/\sigma)^{1/4} - 255 = 1.05705 \sim \lambda(T_o)\Delta R_o = 1.0636$. $1.22*255/288 = 1.08$*]*

Turbulence

Dry static energy at height z and temperature t, g = downward acceleration of gravity:

$s = c_pT + gz$; c_p = specific heat of air at constant pressure = 1,004 $JK^{-1} kg^{-1}$
$s = [m^2s^{-2}] = [J/kg]$....this is per unit mass.

c_pT =enthalpy (symbol H?) measurement of energy in a thermodynamic system. It is the thermodynamic quantity equivalent to the total heat content of a system. It is equal to the internal energy of the system plus the product of pressure and volume. More technically, it includes the internal energy, which is the energy required to create a system, and the amount of energy required to make room for it by displacing its environment and establishing its volume and pressure. I think DR uses the *specific ehthalpy* = h = H/kg. If U is internal energy, H = U + pV, [H] = J, whereas c_pT = [J/kg].

$gz = [m^2/s^2]$; $s/c_p = [T + m^2s^{-2} / \{(kg\, m^2s^{-2}) / T\, kg\}] = [T]$. s/c_p has dimensions T.

L := latent heat of condensation, C condensation rate [g per g condensed per t]
LC = rate at which latent heat is released.
Q_{rad} radiative heating per unit mass.

"thermodynamic energy equation" ⟹ how temperature of a parcel changes: (D/Dt is Langangian derivative, travelling with parcel)

$\rho(D/Dt)(c_pT) = Dp/Dt + LC + Q_{rad.}$

As a parcel moves upward, s is nearly constant. Without external heat source or sink, dry static energy is (nearly) constant:

$\rho Ds/Dt = LC + Q_{rad}$.

A basic state in which the dry static energy increases upward is said to be stably stratified because vertical motions in such a column are resisted by buoyancy. If a parcel in equilibrium is displaced either upward or downward, buoyancy pushes it back toward its starting point. The buoyancy force arises because the parcel conserves its dry static energy as it moves up or down, in the presence of stratification. If the parcel moves upward, it finds itself surrounded by air with a larger dry static energy.

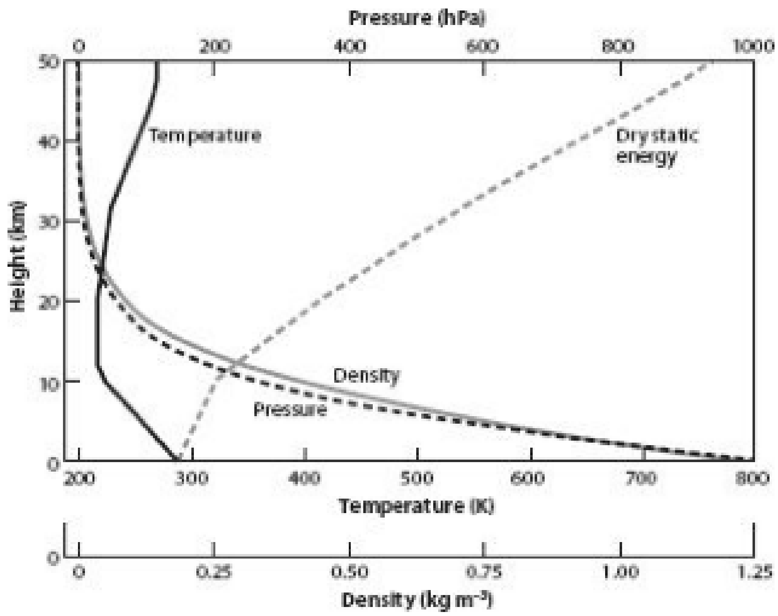
Turbulence can vertically homogenize s.

Cumulus instability is a simple, very important process that occurs quite commonly in many parts of the world. Humid air breaks away from the boundary layer, and floats upward under the influence of the positive buoyancy generated through the release of latent heat, just as a hot air balloon can be lofted by a pulse of heat from its burner.

“mass fraction” = concentration of water vapor = “specific humidity”= q := density water vapor / density = ρ_{vapor}/ρ :

$\rho Dq/Dt = -C$; C is rate at which vapor condenses to liquid.

saturation vapor pressure: The pressure of the vapor in equilibrium with a neighboring liquid surface, denoted by e*(T), is an exponentially increasing function of the temperature. At the globally averaged surface temperature of the Earth, which is 288 K, e*(T) increases at the spectacular rate of 7% per Kelvin.



Saturation specific humidity = q* =value of specific humidity at which partial pressure of water vapor = saturation value, so e = e*(T). ε = molecular wgt of water vapor / molecular wgt of mixture. ε = 0.622.

$q^* = \varepsilon e^*(T)/p$; (ideal gas law)

we write q*(T,p) because q* depends on T and p.

Lapse rate $= \Gamma = -\partial T/\partial z$

If $\delta s/\delta z = 0$ then lapse rate $= \Gamma_{dry} = g/c_p$ = “dry adiabatic lapse rate”
$= 9.8 ms^{-2} / (1004\ JK^{-1} kg^{-1}) = (J=kgm^2s^{-2}) = 9.8K/1004m \sim 10K/m$.

“*moist static energy”* = h := s + Lq = dry static energy + latent energy.

Recall: $\rho Dq/Dt = -C;$
$\rho Ds/Dt = LC + Q_{rad}.$

$L\rho Dq/Dt = -LC \Longrightarrow L\rho Dq/Dt + \rho Ds/Dt = \rho Dh/Dt = Q_{rad} + \rho q DL/Dt$ (I guess last term is small, or zero?). SO, h is driven by Q_{rad}, h is conserved under evaporation / condensation.

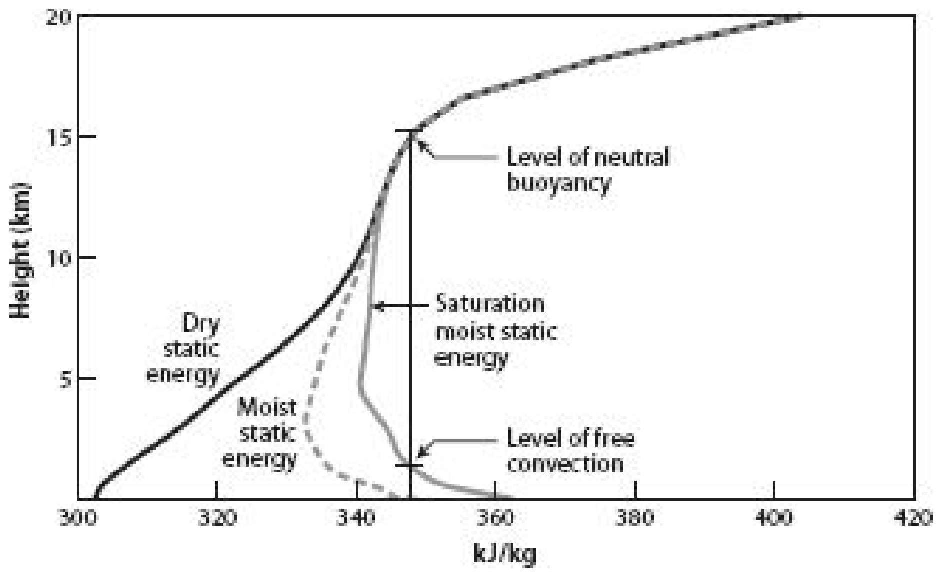


Figure 3.5. The observed vertical distribution of the moist static energy (dashed curve), for January, in kJ kg^{-1}.
A minimum occurs in the tropics, about 5 km above the surface. The vertical profiles of the dry static energy (solid black curve) and saturation moist static energy (solid gray curve) are also shown. The thin vertical line represents the moist static energy of a parcel rising moist adiabatically from near the surface, conserving its moist static energy. The parcel is positively buoyant whenever the thin vertical line is to the right of the dashed line, that is, from about 1 km to 15 km above the surface. Further explanation is given in the text. The plots are based on time averages of data collected during a field experiment called TOGA COARE.

FEEDBACKS

Snow and Ice Feedback

$\Delta a = (\partial a/\partial T)\Delta T$, solve:

$$\left(\frac{\Delta T_S}{T_0}\right) \cong \frac{\left(\frac{\Delta S}{S_0}\right) - 4\frac{\Delta \varepsilon}{\varepsilon_0}}{1 + \left(\frac{16T_0}{1-\alpha_0}\right)\left(\frac{\partial \alpha}{\partial T_S}\right)_{\mathrm{ice}}}.$$

Water vapor feedback

The saturation vapor pressure of water increases exponentially with temperature. There is a tendency for the relative humidity of the air to remain approximately constant as the climate changes. Water vapor content of the atmosphere will increase by about 7% per K as the climate warms. Because water vapor is a powerful greenhouse gas, that is, it strongly absorbs and emits infrared radiation, an increase in the atmospheric water vapor content causes an increase in the downwelling infrared radiation at the Earth's surface. This favors a further warming of the oceans. The initial perturbation is amplified. Surface evaporation can moisten a deep layer of air only if a mechanism exists to carry water vapor upward away from the surface. Boundary-layer turbulence helps, but only through the depth of the boundary layer, which is typically less than 1 km. The most important mechanism for such further lifting is cumulus convection. The mass of water vapor in the atmosphere exceeds the mass of carbon dioxide by about a factor of 4, and water vapor contributes more to the downward infrared at the Earth's surface than CO2 does. Despite these facts, CO2 plays a controlling role in the greenhouse effect, while water vapor plays a subservient role. The reason is that water vapor is condensible and can be removed from the atmosphere by precipitation, whereas CO2 does not condense under conditions found in the Earth's atmosphere. Lacis et al. (2010) illustrated the primary role of CO2 through a clever and simple experiment with a climate model. When run with realistic present-day CO2 concentrations, the model produces a realistic simulation of today's observed climate. In the experiment, Lacis and colleagues removed all of the CO2 (and other noncondensing greenhouse gases) from the atmosphere. The results were spectacular. The surface temperature began to fall immediately as a direct result of the absence of CO2. This led to a reduction in the water vapor content of the atmosphere and additional cooling through the water vapor feedback. The positive snow and ice albedo feedback also reinforced the cooling. Within 10 simulated years, the Earth approached an ice-covered state in which the frigid model atmosphere contained very little water vapor.

Combining Feedbacks

bulk emissivity: **$\varepsilon = \varepsilon_{CO2} + \varepsilon_{H2O}$ (using 'wrong'formula in (5.1)):**

$$\left(\frac{\Delta T_S}{T_0}\right) \cong \frac{\left(\frac{\Delta S}{S_0}\right) - 4\frac{(\Delta \varepsilon)_{CO2}}{\varepsilon_0}}{1 + \left(\frac{16T_0}{1-\alpha_0}\right)\left(\frac{\partial \alpha}{\partial T_S}\right)_{\mathrm{ice}} + \left(\frac{4T}{\varepsilon_0}\right)\frac{\partial \varepsilon_{H2O}}{\partial T_S}}. \qquad (5.5)$$

compare (2.14)

$\Delta T/T = \frac{1}{4}[(\Delta S/S) - \Delta a/(1-a) - \Delta\varepsilon/\varepsilon]$.

Low cloud feedback

Optically thick clouds behave like blackbodies. Low clouds emit at relatively warm temperatures, not very different from the temperature of the Earth's surface. As a result, an increase in low cloud amount has relatively little effect on the OLR. On the other hand, low clouds can be very bright (as seen from above), reflecting back to space up to half of the solar radiation that hits them. As a result, they tend to cool the Earth, in the present climate. In a future climate warmed by increasing greenhouse gases, an increase in low cloud amount would increase the cooling, and so could reduce the warming. On the other hand, a decrease in low-cloud amount would increase the warming. Low cloud amount tends to be greater when the sea-surface temperature (SST) is cooler than the average SST at that latitude. Over cold water, low clouds typically take the form of a uniform overcast. When the water is warmer, the low clouds change their type, to shallow cumulus clouds with smaller cloud fractions.

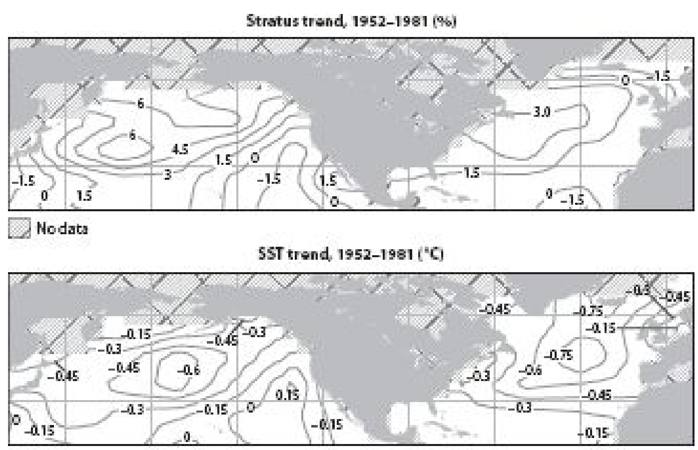


Figure 5.3. Observed trends of stratus cloud amount (top) and sea surface temperature (bottom), for the period 1952– 81. Source: The figure was provided by Joel Norris. It is based on work published by Norris and Leovy (1994).

Figure 5.3 shows a possible example of an observed (Norris and Leovy, 1994) cloud feedback. The two panels of the figure show observed trends in SST and stratocumulus cloudiness, over a 30-year period. Downward trends in sea surface temperature are colocated with upward trends in stratocumulus amount, and vice versa. There are at least two possible interpretations of these correlated trends in SST and stratocumulus cloud amount, which do not contradict each other. The first is that a cooling (warming) of the sea favors an increase (decrease) in low cloud amount; this is plausible in light of our understanding of the physics of low-level marine clouds. The second interpretation is that an increase (decrease) in stratus cloud amount favors a decrease (increase) in the sea surface temperature because the clouds reflect solar radiation that would otherwise be absorbed by the ocean. The positive cloud feedback suggested in Figure 5.3 is called a shortwave cloud feedback because it primarily involves solar radiation. Clement et al.

(2009) provide further observational evidence for a positive shortwave cloud feedback associated with low clouds.

High Cloud Feedback

High, cold cirrus clouds are often somewhat transparent, so they don't reflect as much solar radiation as typical low-level clouds, but they do efficiently absorb the infrared radiation coming from the Earth's surface, leading to a radiative warming near the cloud-base level. The high clouds emit to space, but only weakly because of their very cold temperatures. They therefore tend to warm the Earth as a whole. An increase in high cloud amount will tend to enhance greenhouse warming, while a decrease will tend to reduce it. Even if the high-cloud amount remains the same, the high clouds can produce a positive feedback as the surface warms up, because they will absorb the increased infrared coming from the warmer surface. Tropical cumulus clouds often stop about 15 km above the Earth's surface, where the temperature is about 200 K. Hartmann and Larson (2002) suggested that the reason is that at a temperature of 200 K the infrared radiative cooling of the atmosphere becomes weak. The cooling becomes weak because the water vapor concentration is very small. The air is dry because it is so cold that the saturation specific humidity of water vapor is very small. Radiative cooling aloft tries to steepen the lapse rate and generates gravitational potential energy. Warming produced by the cumulus clouds balances the radiative cooling. When the radiative cooling stops, the cumulus cloud warming must also become weak. According to Hartmann and Larson, this is why the cumulus clouds top out at the level where the temperature is 200 K. Taking this idea a step further, Hartmann and Larson hypothesized that if the climate changes in such a way that the height where the temperature is 200 K moves up or down, the tops of the cumulus clouds will move up and down too, so that the cloud-top temperature is always 200 K. This is called the “fixed anvil temperature” (FAT) hypothesis. The tall tropical cumuli produce broad regions of optically thick cirrus and anvil clouds near their tops, which therefore emit infrared at a temperature of about 200 K. The FAT hypothesis of Hartmann and Larson implies that this will be true even in a different climate. It follows that in regions of deep cumulus convection, the OLR will not change even when the climate changes. The OLR will therefore be insensitive to the surface temperature. The Earth will not be able to increase its infrared emission to space as the surface temperature increases. The bulk emissivity will therefore decrease as the Earth warms. This is a positive cloud feedback. It is called a longwave cloud feedback because it primarily involves infrared radiation. Further discussion is given by Zelinka and Hartmann (2010).

The Lapse-Rate Feedback

A climate change in which the warming increases with height is one in which the lapse rate decreases. The change in the lapse rate is a feedback. An infrared photon that is emitted in the upper atmosphere can escape to space more easily than one that is emitted near the surface. Suppose that the atmosphere warms up, so that the overall rate of emission increases. If the warming is mostly in the upper troposphere, the additional thermal energy be radiated away to space easily, and the bulk emissivity increases. If the warming is mostly in the lower troposphere, the energy has a harder time getting out to space, and the bulk emissivity decreases. For a very simple reason, explained below, there is a tendency for the lapse rate to become weaker as the surface temperature warms. This is especially true in the tropics. If the surface temperature warms, the lapse rate decreases, and the upper tropospheric temperature warms even more. The bulk emissivity increases, and the warming is damped. This is a negative longwave feedback. The reason why the lapse rate decreases as the temperature warms can be seen in Figure 3.5, which shows the variations of the moist adiabatic lapse rate with temperature and pressure. For a given pressure, the moist adiabatic lapse rate decreases as the temperature increases. This follows directly from the thermodynamic properties of water. As discussed in Chapter 3, the actual lapse rate of the tropical troposphere, and also to some extent the summer midlatitude troposphere, approximates the moist adiabatic lapse rate. If the surface temperature warms, the lapse rate decreases, and the upper troposphere warms even more. The lapse rate feedback is negative because it tends to damp changes in the surface temperature. It is important to realize that the feedbacks themselves are manifestations of climate change. Even a negative feedback, like a decrease in the lapse rate, is a climate change.

From Soden and Held: (from RFFDP 16-19)

We use the equations from Soden et al. (2008) to relate decadal change in CRF to equilibrium climate sensitivity (ECS) as defined in IPCC (2013). Let R_f denote the total anthropogenic radiative forcing of climate change by greenhouse gases, aerosols, and land change. T_s is global average surface temperature, and λ is climate sensitivity. Following Soden et al. (2008):

$$\Delta R_f/\Delta T_s = \lambda = \lambda_p + \lambda_L + \lambda_w + \lambda_\alpha + \lambda_{csw} + \lambda_{clw}. \qquad (1)$$

Note $\Delta R_f/\Delta T_s$ is expressed in units of $Wm^{-2}K^{-1}$. The feedbacks are as follows:
λp = plank temperature feedback (pure σT^4: i.e., no atmosphere) ~ –3.2
λL = temperature lapse rate feedback ~ –0.6
λw = water vapor feedback ~ +1.6
λα = snow and ice surface albedo feedback ~ +0.3
λcsw = shortwave cloud feedback (this is what we vary to get cloud feedback relationship to sensitivity and SW CRF)
λclw = longwave cloud feedback (not given separately in the IPCC report; using Soden and Vecchi 2011, Figure 3 top, and averaging for all 12 of the climate models they used) ~ + 0.35

Positive magnitude is a positive feedback, and negative magnitude is a negative feedback.

We use estimates from the IPCC AR5 report, chapter 9, Figure 9.43, and Table 9.5, CMIP5 mean (red dot in the figures) for everything except the LW cloud feedback, which is not given in the IPCC report. LW cloud feedback is taken from Soden and Vecchi (2011).

$$\lambda = \lambda_p + \lambda_L + \lambda_w + \lambda_\alpha + \lambda_{csw} + \lambda_{clw}. \qquad (2)$$

Solving for λ_{csw} with the values above,

$$\lambda_{csw} = \lambda - (-3.2) - (-0.6) - (+1.6) - (+0.3) - (+0.35) = \lambda + 1.55 \qquad (3)$$

λ is simply related to the equilibrium climate sensitivity *(ECS)*, as used in DICE, where ΔCO_2 denotes a doubling of atmospheric CO_2 concentration:

$$\lambda = \Delta R_f/\Delta T_s = (\Delta R_f \text{ for } \Delta CO_2) / (\Delta T_s \text{ for } \Delta CO_2) = -\,3.7 / ECS. \qquad (4)$$

ECS in this definition is the amount of equilibrium global average surface temperature increase for an anthropogenic radiative forcing equivalent to a doubling of CO_2. The factor 3.7 converts a doubling of atmospheric CO_2 to Wm^{-2} of radiative forcing. See IPCC (2013) for a discussion of the definition of radiative forcing. The idea here is that we set all of the feedbacks except SW cloud feedback equal to their average over the climate models. We then vary SW cloud feedback to obtain the range of climate sensitivity.

Combining (2), (3), and (4), the governing equation for the change in CRF is

$$100\Delta CRF(em, t, ECS)/ 50 = 2\ \lambda_{csw}\ \Delta T(em, t, ECS) =$$
$$= 2[-3.7 / ECS - (\lambda_p + \lambda_L + \lambda_W + \lambda_a + \lambda_{clw})] \times \Delta T. \qquad (5)$$

In this equation, 50 is the global mean value of CRF_{sw} in units of Wm^{-2} and is used to convert the trend in Wm^{-2} into a trend in units of a fraction. A factor *100* converts fractions to percentages, resulting in the factor *2*. We then have the decadal trend in shortwave cloud radiative forcing in units of %/decade. ΔT is determined by emissions scenario *em*, time *t*, and *ECS*. $\Delta T(em, t, ECS)$ is computed from DICE. Hence, the RHS is known and we can compute the theoretical value of $\Delta CRF(em, t, ECS)$ based on the IWGSCC certified DICE model, supplemented with Soden et al. (2008). ΔT represents the "true" global mean temperature change under these assumptions. Parenthetically, we note that Roe Baker adopted by IWGSCC use $ECS = 1.2/(1-f), f \sim Normal(0.62, 0.19^2)$, whereas Soden et al. (2008) used effectively $f \sim Normal(0.62, 0.1766^2)$. This difference is negligible.

Feedback Story (Roe & Baker "Seeing Red" 2009)

Forcing from CO2 concentration change [ppmv] is deduced from radiative transfer codes, to the first order

$F = 5.35\times\ln(CO2/C_o)$ [Wm^{-2}] if $CO2 = 2C_o$, $F = 3.7Wm^{-2}$.

$R[Wm^{-2}] = \sigma T^4$; $\sigma = 5.67\times10^{-8}$ [Wm^{-2}/K^4] (Stefan-Boltzmann black body law)

$\lambda(T) = dT/dR = (4\sigma T^3)^{-1}$.

without feedback:
T = 255K, R = 239.7418; put $\Delta R = 4$,

$\Delta T_o = ((R+4)/\sigma)^{1/4} - 255 = 1.05705 \sim \lambda(T_o)\Delta R_o = 1.0636$.

With feedback $c[Wm^{-2}K^{-1}]$, the fraction $c\Delta T_o$ gets fed back to R.

$\Delta T_1 \sim \lambda(T_o)\ (\Delta R_o + c\Delta T_o)$. Apply feedback to ΔT_1:

$\Delta T_2 \sim \lambda(T_o)\ (\Delta R_o + c\Delta T_1) = \lambda(T_o)\ [\ \Delta R_o + c\lambda(T_o)\ (\Delta R_o + c\Delta T_o)]$

$= \lambda(T_o)[\Delta R_o + c\lambda(T_o)(\Delta R_o + c\lambda(T_o)\Delta R_o) = \lambda(T_o)\Delta R_o[1+ c\lambda(T_o) + (c\lambda(T_o))^2]$,

$\lim_i \Delta T_i = \mathrm{Lim}_i\ \lambda(T_o)\ \Delta R_o\ (\Sigma_{i=0..\infty}\ (c\lambda(T_o))^i) = \lambda(T_o)\ \Delta R_o / (1- c\lambda(T_o))$.

Roe does this in one step by solving the following for ΔT:

$\Delta T \sim \lambda(T_o)\ (\Delta R_o + c\Delta T)$.

However, the geometric series is better for removing the error caused by evaluating the derivative 'at the wrong value of T'.

Indeed, note that $\lambda(T_o)$ is the derivative evaluated at T_o whereas the derivative at $T^* = [(R_o + c\Delta T_o)/\sigma]^{1/4}$ is $1/(4T^{*3}) = (4[(R_o + c\Delta T_o)/\sigma]^{3/4})^{-1}$. $\lambda(T_o) = 0.265911$; $\lambda(T^*) = 0.262631$.

If we make this correction we no longer get an analytically summable geometric series, but the resulting series converges quickly:

$\Delta T_1 \sim \lambda_1 (\Delta R_o + c\Delta T_o)$; $\lambda_1 = (4[(R_o + c\Delta T_o)/\sigma]^{3/4})^{-1}$.

$\Delta T_2 \sim \lambda_2 (\Delta R_o + c\Delta T_1) = \lambda_2\Delta R_o + \lambda_2 c\, \lambda_1 (\Delta R_o + c\Delta T_o) = \lambda_2\Delta R_o + \lambda_2 c\, \lambda_1\Delta R_o + \lambda_2 c\, \lambda_1 c\lambda_o\Delta R_o)$

$= \Delta R_o[\lambda_2 + c\lambda_2\lambda_1 + c^2\lambda_2\lambda_1\lambda_o]$; $\lambda_2 = (4[(R_o + c\Delta T_1)/\sigma]^{3/4})^{-1}$

ΔT_n is a good estimate for n = 5.

A Better Way

This is after all a pain in the posterior. A better way is to notice that we seek ΔT that solves

$$\frac{(R + \Delta R + c\Delta T)^{1/4}}{\sigma^4} - \frac{R^{1/4}}{\sigma^4} = \Delta T.$$ Excel's solver does this in a wink.

Guess a value for the RHS, substitute it in the LHS, compute the squared error, and minimize.

Even Better, guess ΔT in LHS, compute LHS and substitute result in LHS, repeat. It converges like a Bat out of Perdition. It is numerically identical to the optimization result.

Here's some results.

			Delta T		
			Roe feedback	Real feedback	Optimization & iteration
T=255	Delta R = 1	c=.05	0.269494	0.268631	0.269062246
		c=.1	0.273175	0.272277	0.272725296
		c=.5	0.306687	0.305417	0.306050786
		c=.9	0.34957	0.347693	0.348629437
	Delta R = 4	c=.05	1.077977	1.064327	1.071117658
		c=.1	1.0927	1.07849	1.085558785
		c=.5	1.226747	1.20673	1.216672236
		c=.9	1.398281	1.368834	1.383427139