Tail Dependence and Vines

Cooke 5/19/2015

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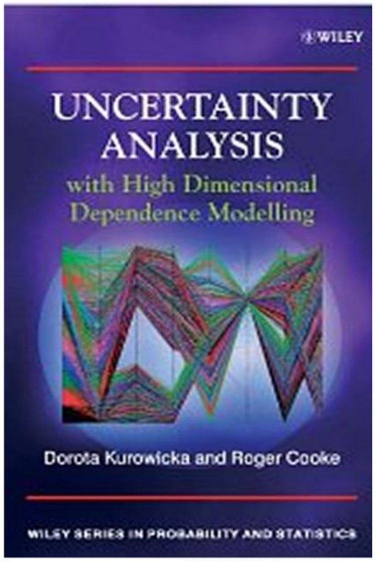
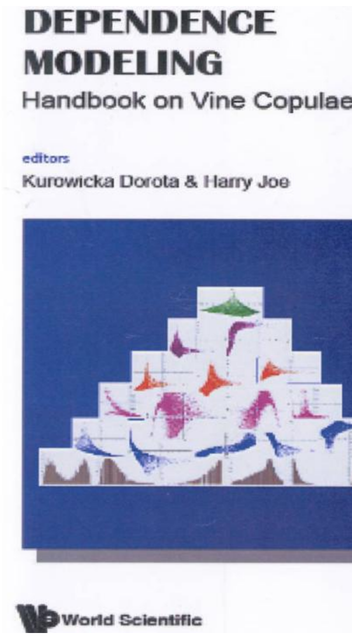




Copulae and Tail Dependence

Copula: $C(u_1, u_2) = P(U_1 \le u_1, U_2 \le u_2)$; $U_i \sim Unif(0,1)$.

$\partial_2 C(u_1, u_2) = \lim_{\delta \to O} [\, P(U_1 \le u_1, U_2 \le u_2 + \delta) - P(U_1 \le u_1, U_2 \le u_2)\,] / \delta$

$= P(U_1 \le u_1, U_2 = u_2) = P(U_1 \le u_1 \mid U_2 = u_2)$ (since U_2 uniform).

Lower Tail dependence: $\lambda_l = \lim_{q \to 0+} C(u, u)/u$.

NB for $C(u_1, u_2) = 1$ (independent),
$\lim_{q \to 0+} C(u, u)/u = q^2/q \to 0$. **Upper TD** defined similarly for $u^* = 1-u$.

L'Hôpital:
$\lim_{q \to 0+} C(u, u)/u = \lim_{q \to 0+} dC(u, u)/du = P(U_1 \le u_1 \mid U_2 = u_2) + P(U_2 \le u_2 \mid U_1 = u_1)$.

C symmetric $\Rightarrow \lambda_1 = 2 \lim_{q \to 0+} P(U_1 \le q \mid U_2 = q)$.

Normal Copula, Tail Independence

$2 \lim_{q \to 0+} P(U_1 \le q \mid U_2 = q) = 2 \lim_{x \to -\infty} P(X_1 \le x \mid X_2 = x)$, $X_i \sim N(0, 1)$.

$Y := (X_2 \mid X_1 = x) \sim N(\rho x, 1-\rho^2)$.

$P(Y \le x) = \Phi((x-\rho x)/\sqrt{(1-\rho^2)}) = \Phi(x\sqrt{(1-\rho)}/\sqrt{(1+\rho)}) \to 0$; $|\rho| < 1$.
[NB $\Phi(x\kappa)/\Phi(x) \sim \phi(x\kappa)/\phi(x) \to \infty$ as $x \to -\infty$.]

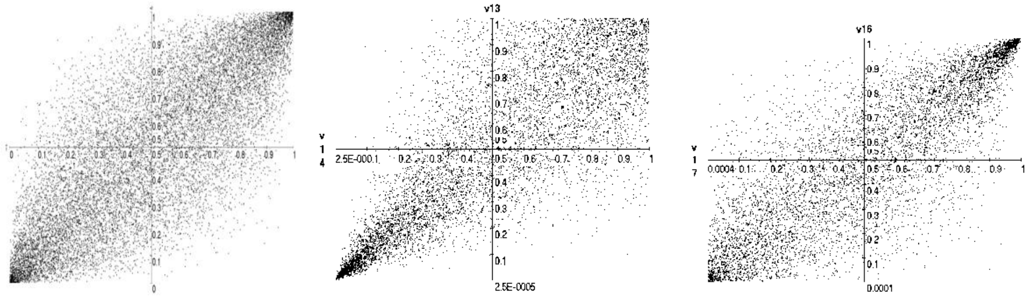
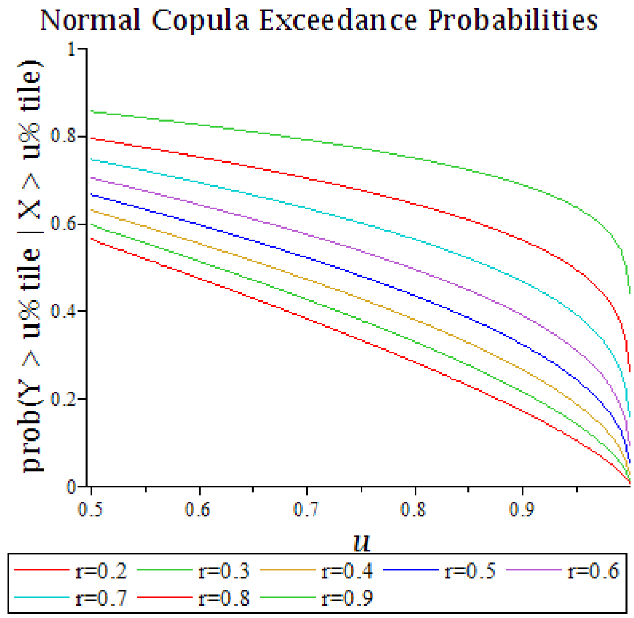


Figure 1: Normal (left) , Clayton (middle) and Gumbel (right) copula, correlation 0.8.

The following graph shows the probability that Y exceeds its u-th quantile given that X exceeds its u-th quantile, u = 0.5 ... 0.99, for correlation values r = 0.5 and r = 0.9.



All limiting exceedance probabilities are zero

Figure 2: Conditional exceedance for the normal copula, correlation = 0.5 … 0.9.

We see that for rank correlation r = 0.5 (blue), the probability that Y exceeds its median given that X exceeds its median is about 0.6667. The probability that Y exceeds its 0.99 quantile given that X exceeds its 0.99 quantile is 0.13. For rank correlation r = 0.9, the same exceedance probabilities are 0.856 and 0.543. At some point the conditional exceedance probability dives down in a supra linear fashion. The behavior for upper and lower exceedance probabilities is the same, owing to the symmetry of the normal copula. Note that these exceedance probabilities do NOT depend on the distributions for X and Y, since the exceedances are couched in terms of percentiles.

A tail dependent copula will show very different behavior. The Gumbel copula is the most popular simple copula for capturing upper tail dependence. The conditional exceedance

The Gumbel copulae with correlation 0.8 is shown in Figure 1. P(Y > u-th quantile | X > u-th quantile) converges to a constant, which parameterizes the Gumbel family[1].

The conditional exceedance probabilities for the Gumbel copula are shown in Figure 3, which should be compared with Figure 2. At the median (u = 0.5) the conditional probabilities are comparable to those in Figure 2. However, at the 99th percentiles, for r=0.5 and r=0.9 these are respectively 0.436 and 0.797 respectively.

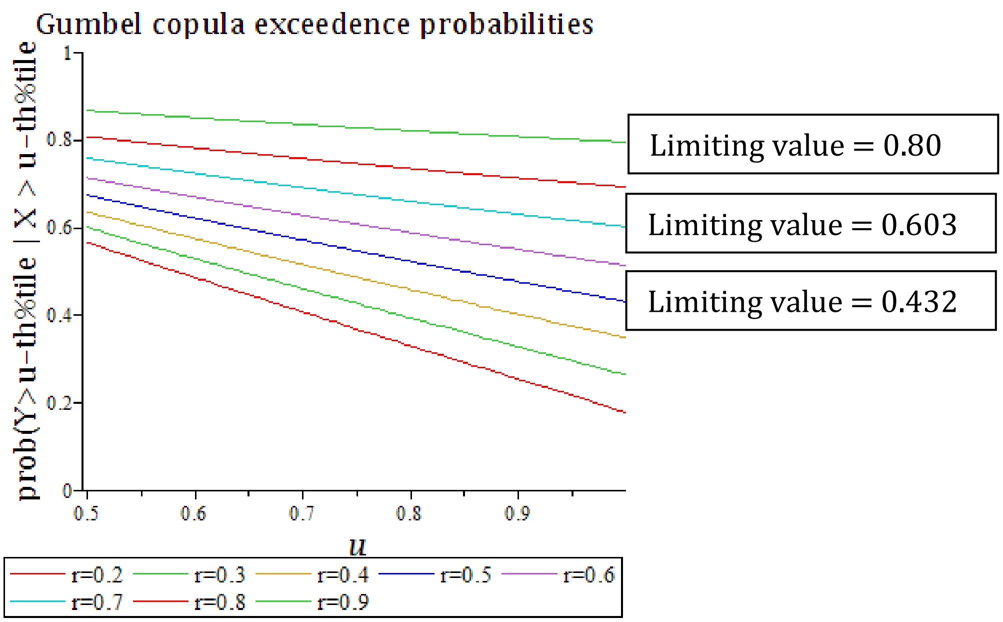


Figure 3 Conditional exceedance for the Gumbel copula, correlation = 0.5 … 0.9.

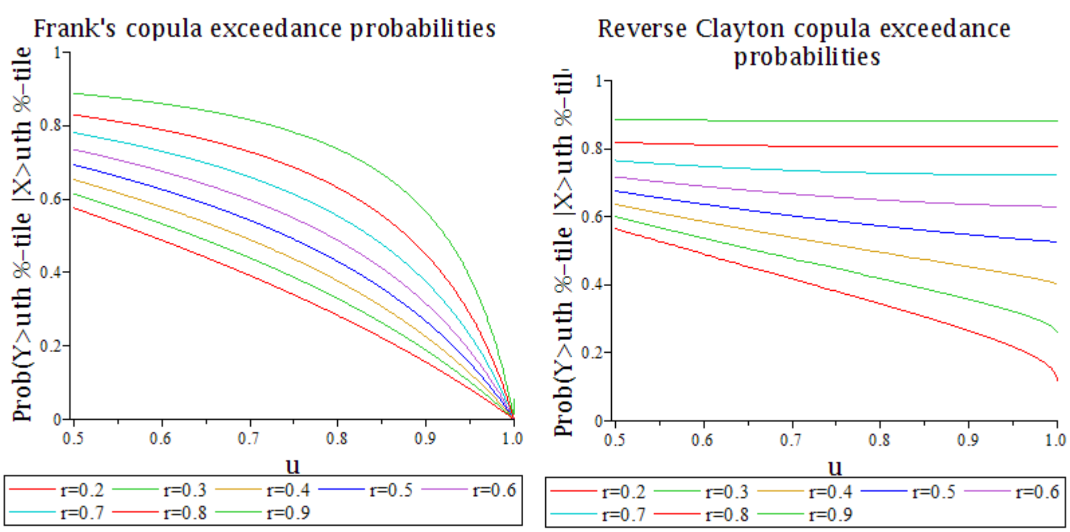
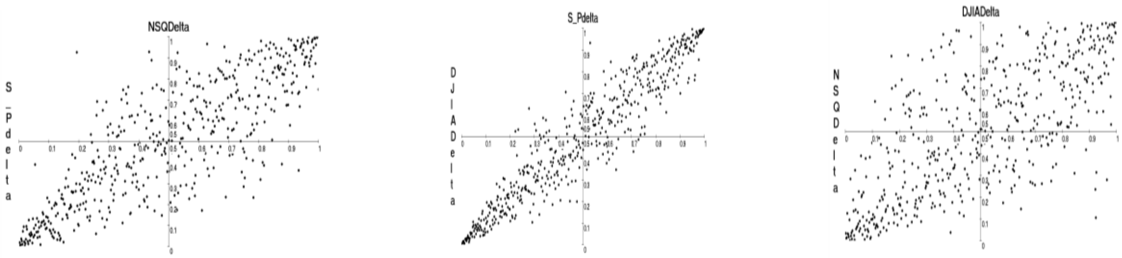


Figure 4 Conditional exceedance for Frank and Reverse Clayton, correlation = 0.5 … 0.9.

[1] P(Y≤ u-quantile ∩ X ≤ v-quantile) = $\exp\{-((-\ln(u))^{\alpha} + (-\ln(v))^{\alpha})^{1/\alpha}\}$, where $\alpha \geq 1$. An elementary computation shows that $\lim_{u\to 1}$ P(Y > u-quantile | X > u-quantile) = $2 - 2^{1/\alpha}$.

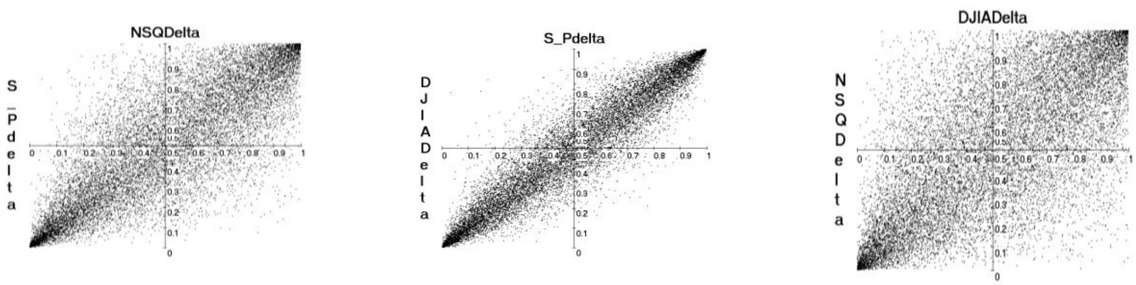
Delta = closing price – next day open price

last 2 years



Fit Joint with Regular Vine 15,000 samples

Student t survivor Gumbel complex



Aggregation Amplifies Dependence

Micro-correlations are correlations between variables at or beneath the limit of detection. The difficulty with micro-correlations is that they could so easily go undetected. One might not readily assume that fires in Australia and floods in California are correlated, for example, but El Niño events induce exactly this coupling. These tiny correlations are amplified by aggregation, undermining common diversification strategies.

The amplification under aggregation is illustrated by a very simple formula that should be on the first page of every insurance text book, but isn't. Let $X_1,..X_N$ and $Y_1, \ldots Y_N$ be two sets of random variables with the same average variance σ^2 and average covariance C (within and between sets). The correlation of the sums of the X's and the sum of the Y's is easily found to be:

$$\rho(\sum X_i\,, \sum Y_i) = \frac{N^2 C}{N\sigma^2 + N(N-1)C}\,.$$

This evidently goes to 1 as N grows, *if* C is non-zero and σ^2 is finite. If all variables are independent, then $C = 0$, and the above correlation is zero. The variance of ΣX_i is always nonnegative; if the σ^2 and C are constant for sufficiently large N, it is easy to see that $C \geq 0$.

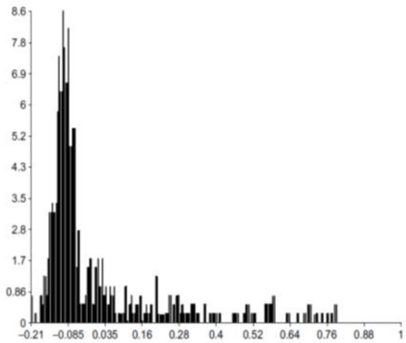


Figure 5.1. Histogram of 500 correlations of randomly paired US exposure-adjusted flood loss per county, 1980–2006. The average correlation is 0.04.

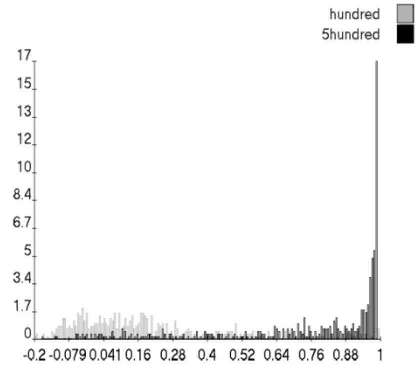
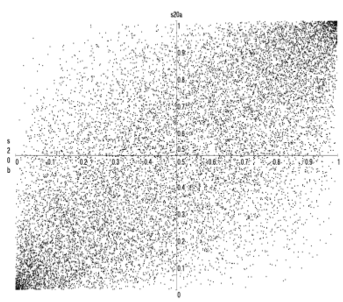


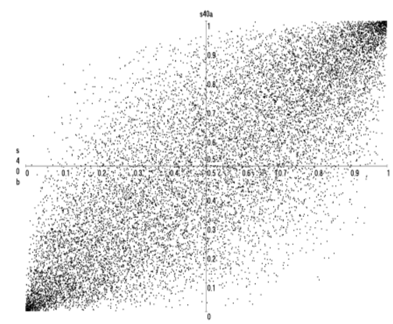
Figure 5.2. Similar to Fig. 5.1, but showing 500 correlations of random sums of 100 and 500.

Amplifying Tail Dependence?

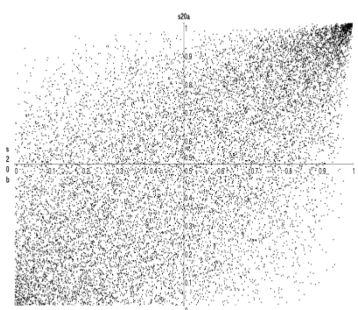
An important question is When is tail dependence amplified by aggregation? Simple simulations show that the answer depends on the copula and the margins. In the graphs below, variables are are conditionally independent given a latent variable, to which they are all weakly coupled. For the rest, the little that is known is given below.

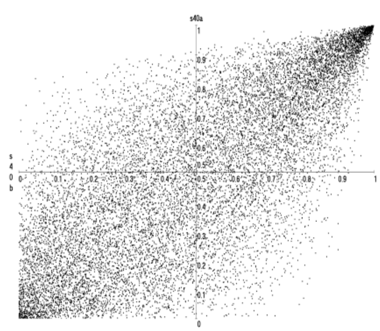
normal, normal copula 0.3 to Latent, sums of 20 sums of 40



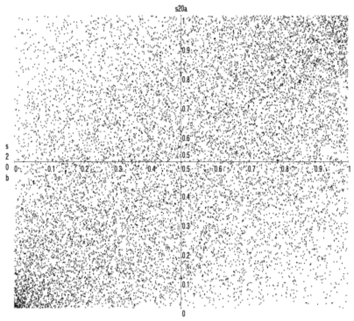


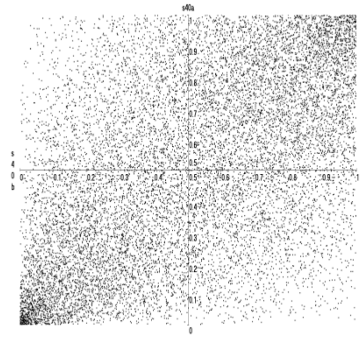
normal, Gumbel copula 0.3 to Latent, sums of 20 sums of 40



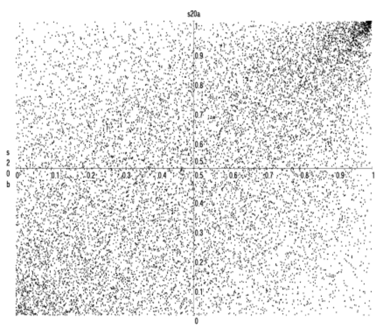


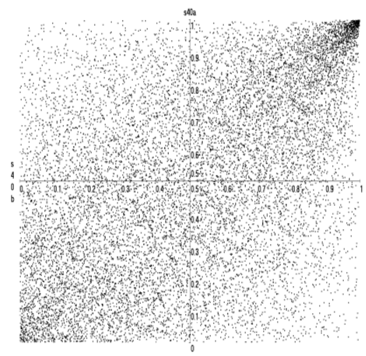
Pareto(1), normal copula 0.3 to Latent sums of 20 sums of 40





Pareto(1), Gumbel copula 0.3 to Latent sums of 20 sums of 40



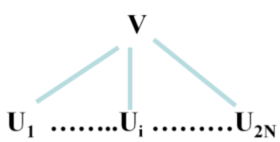


How does tail dependence arise?

1) Aggregating events
2) Aggregating events + damages
3) Scale mixtures

Aggregating events

Latent Variable Model For Sums of Events



$Y_i = 1_{\{U_i > r\}}$;
$S_1(N) = Y_1 .. +.. Y_N$; $S_2(N) = Y_{N+1} .. +.. Y_{2N}$

Under what conditions does $UTD(S_1(N), S_2(N)) \to 1$?

Harry Joe's UTD condition:

$P\{U \leq u \mid V=1\} = 0$, $\forall\ 0 < u < 1$.

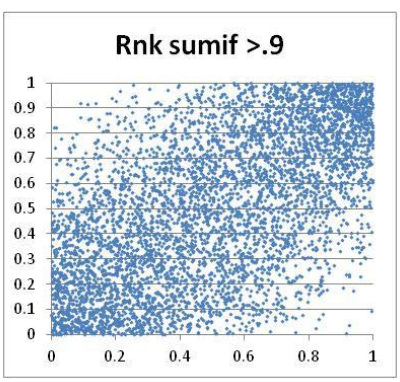
FRANK: No
Standard UTD copulae: YES
NORMAL: YES (!)

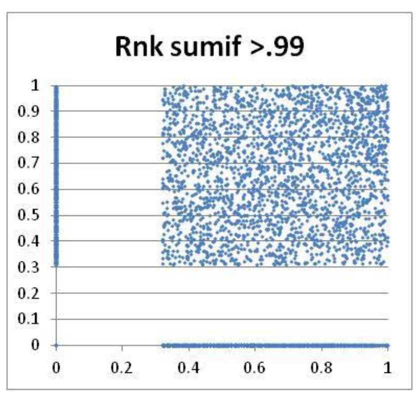
Table 5.1. Conditional probabilities $\Pr(S_2 > N\zeta | S_1 > N\zeta) = \lambda_U(r, \zeta, N)$ with $r = 0.9$, $\zeta = 0.7$, Spearman ρ_S = rank correlation = 0.5; leading to parameters $\theta = 1.54$ for the Gumbel, $\rho = 0.518$ for the bivariate normal (BVN), $\theta = 7.90$ for the Frank copulae respectively. Limit behavior depends on the comparison sign of $p_r(1) - \zeta$.

	$\lambda_U(r, \zeta, N)$		
N	Gumbel	BVN	Frank
10	0.604	0.264	0.144
20	0.687	0.411	0.067
30	0.733	0.528	0.034
40	0.763	0.620	0.019
50	0.785	0.695	0.010
60	0.801	0.755	0.006
70	0.815	0.804	0.003
80	0.826	0.845	0.002
90	0.835	0.877	0.001
100	0.843	0.903	0.001

256 vbls, correlation to Latent = 0.2

Sums of 125 disjunct FRANK





Aggregating events and damages

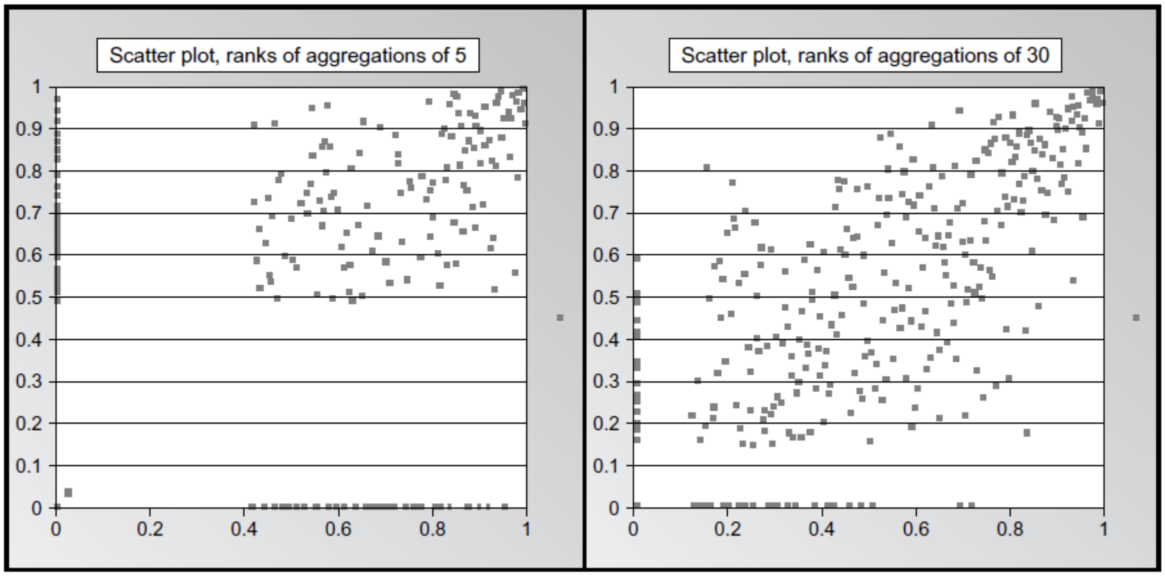


Figure 5.5. Percentile scatterplots of random aggregation of Florida county monthly flood losses. Left: two random aggregations of five counties; right: two random aggregations of 30 distinct counties.

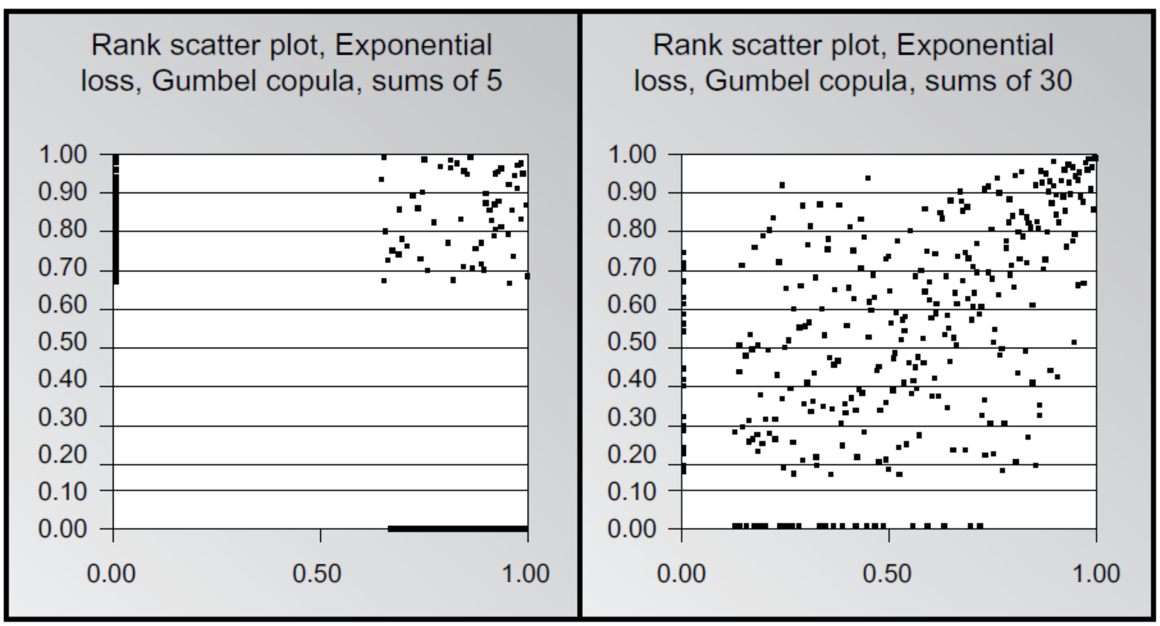


Figure 5.8. A model for Florida monthly flood damages, exponential damages linked to a latent variable with the Gumbel copula.

Scale mixtures

Mixtures of exponentials: Consider exponentials conditionally independent given rate, with gamma mixing of rates. Marginal distributions are Pareto with parameter depending on shape of gamma. Sums are positive tail dependent:

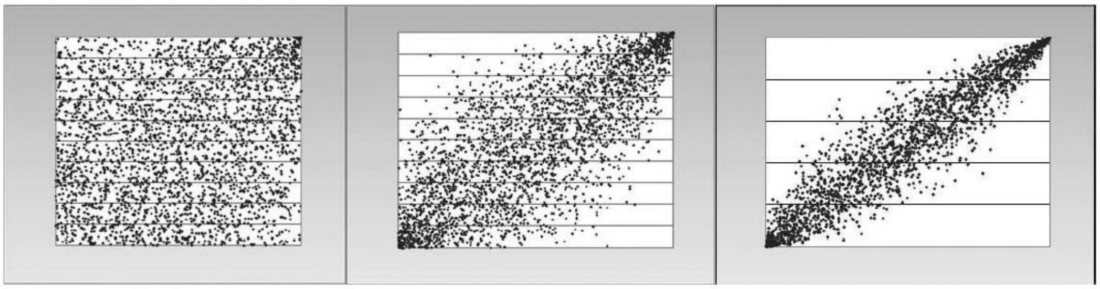
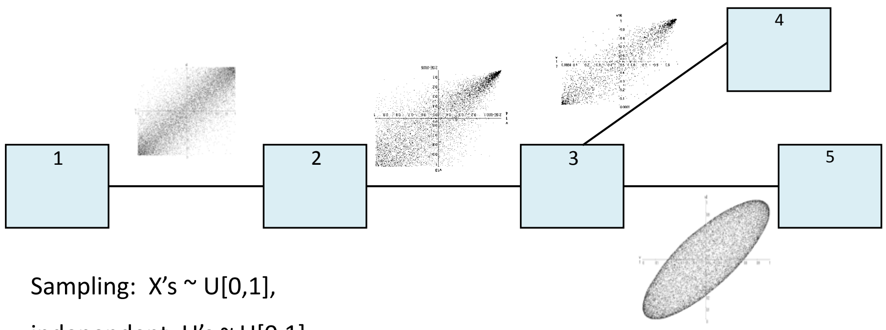


Figure 5.11. Percentile scatterplots for sums of L_1 variables, with shape of Gamma mixing distribution = 3. Left: 2 L_1 variables, rank correlation = 0.21; center: sums of 10 such variables, rank correlation = 0.77; right: sums of 50 such variables, rank correlation = 0.94.

VINES

Markov trees:



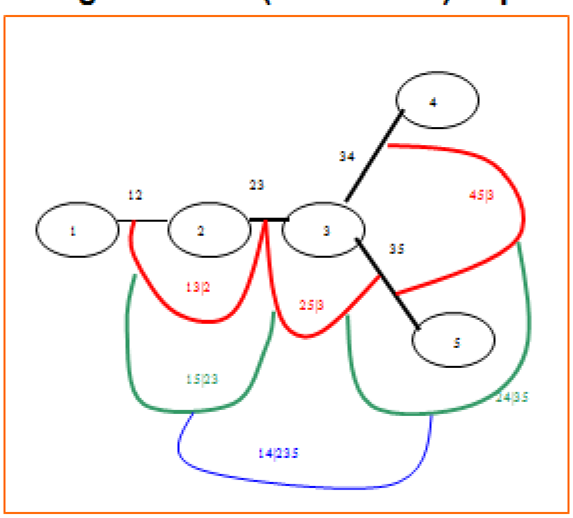
Sampling: X's ~ U[0,1],

independent, U's ~ U[0,1]

$$x_1 = u_1$$
$$x_2 = F^{-1}_{2|1:x_1}(u_2)$$
$$x_3 = F^{-1}_{3|2:x_2}(u_3)$$
$$x_4 = F^{-1}_{4|3:x_3}(u_4)$$
$$x_5 = F^{-1}_{5|3:x_3}(u_5)$$

Regular Vine:
System of Bivariate Constraints

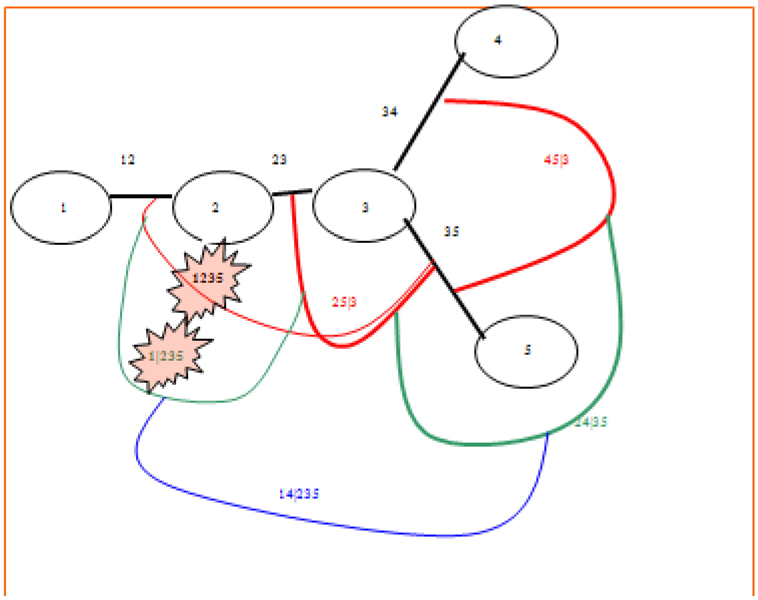
Regular vine + (conditional) copula => sampling algorithm

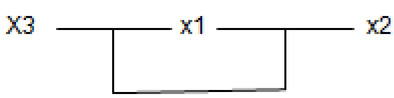


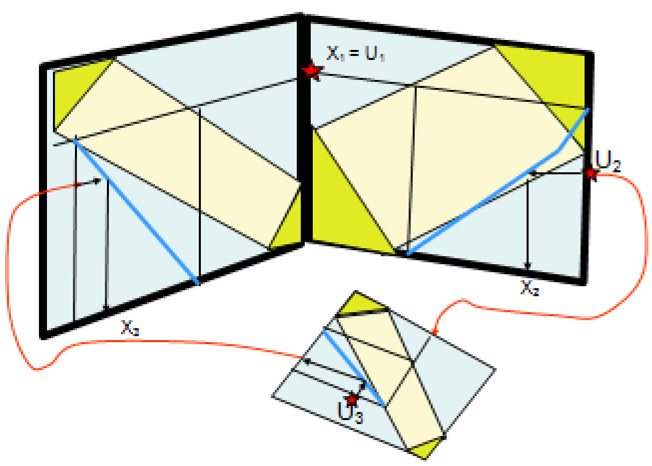
Constraints on edges: Δ | ∩

- Δ = doubleton
- Every pair occurs once as Δ,
- Copulae can be chosen arbitrarily, typically indexed by (rank) correlation
- Setting ij | K = Independent => max. Inf realization given rest

Non Regular Vine







Theorem 3.2 *Let D be the determinant of the n-dimensional correlation matrix ($D > 0$). For any partial correlation vine*

$$D = \prod_{e \in E(V)} (1 - \rho^2_{e_1,e_2;D_e}) \tag{4}$$

Why: where R is *multiple correlation:*

$$D = (1-R^2_{1\{2..n\}})(1-R^2_{2\{3..n\}})\ldots(1-R^2_{n-1\{n\}}).$$

Where ρ is partial correlation:

$$(1-R^2_{1\{2..n\}}) = (1-\rho^2_{12;3..n})(1-\rho^2_{13;4..n})\ldots(1-\rho^2_{1n})$$

The search for an optimal vine requires a method for enumerating and searching all vines. The number of regular vines grows very quickly. A closed formula for the number of regular vines on n elements was found in Morales Napoles et al.[41] :

Theorem 3.1.

(1) For any regular vine on $n-1$ elements, the number of regular n-dimensional vines which extend this vine is 2^{n-3}.

(2) There are $\binom{n}{2} \times (n-2)! \times 2^{(n-2)(n-3)/2}$ labeled regular vines in total.

Note that the number of extensions of a regular vine does not depend on the vine itself.

N	N!	#reg Vines
2	2	1
3	6	3
4	24	24
5	120	480
6	720	23040
7	5040	2580480
8	40320	660602880
9	362880	3.80507E+11
10	3628800	4.87049E+14
11	4E+07	1.37153E+18
12	4.8E+08	8.42669E+21
13	6.2E+09	1.12176E+26
14	8.7E+10	3.21631E+30
15	1.3E+12	1.9761E+35
16	2.1E+13	2.59012E+40
17	3.6E+14	7.2142E+45
18	6.4E+15	4.25511E+51
19	1.2E+17	5.29839E+57
20	2.4E+18	1.38894E+64

Regular Vine + Conditional Copulae

Theorem [1] Let $\mathcal{V}$ be a regular vine on n elements, for each edge e $\in E(\mathcal{V})$, let the conditional copula and copula density be $C_{e1,e2 \mid De}$, $c_{e1,e2 \mid De}$, let 1-D margins with cdf F_i and densities f_i be given. Then the vine-dependent distribution is uniquely determined and has density:

$$f_{1,\ldots n} = f_1 \ldots f_n \, \Pi_{e \in E(\mathcal{V})} \, c_{e1,e2 \mid De} \, (F_{e1 \mid De}, F_{e2 \mid De}).$$

Any joint density can be represented in this way, for any regular vine….conditional copula NOT constant

Compare Hammersley Clifford

Tail dependent copula often arise in fitting financial time series.

In the example below, N dentoes the normal copula, F is the Frank copula, t is the T copula and G is the Gumbel copula.

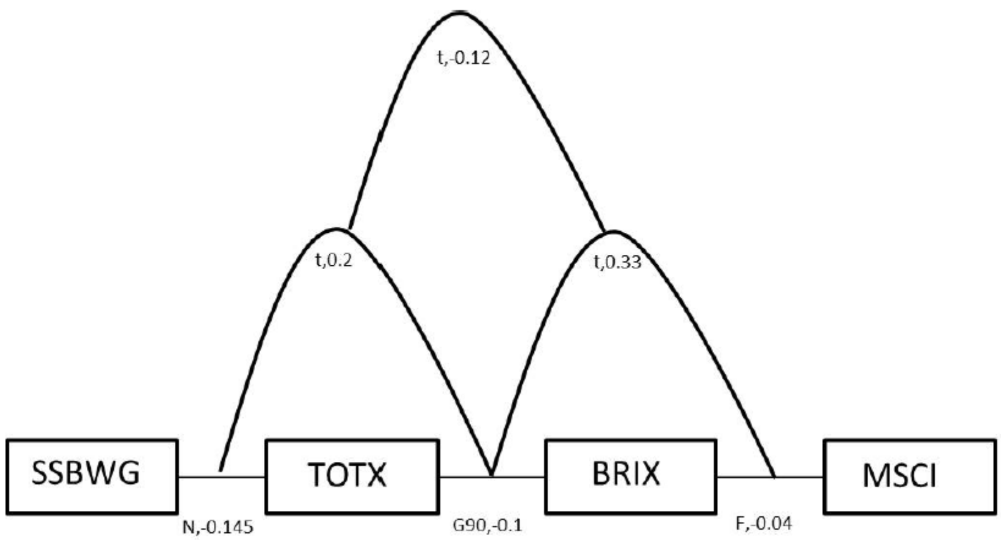
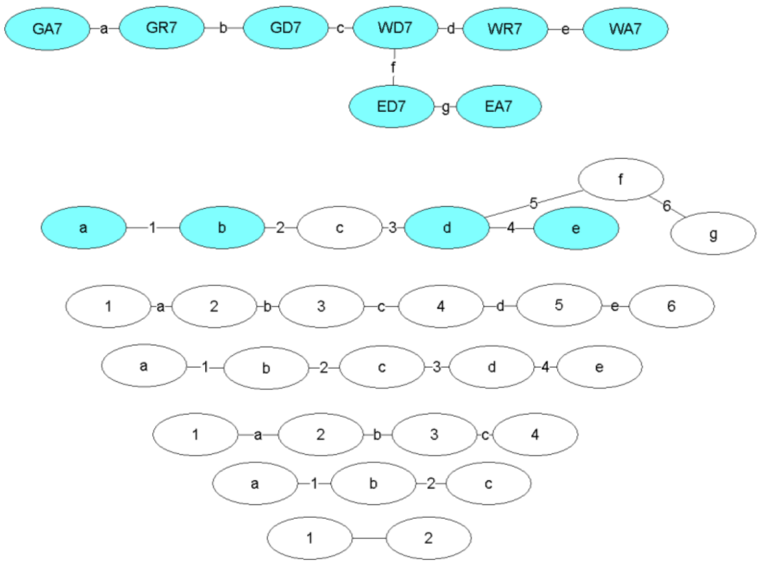


Figure 12: The D-vine fitted to the Norwegian stock market data.

Ice Sheet Elicitation (16 experts)
Accumulation, Discharge, Runoff in
Greenland, West Antarctica, East Antarctica



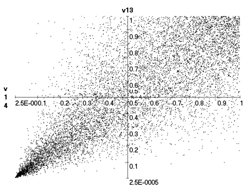
	Mean	Stdev	5%	50%	95%
EW slr indep	3.48E+02	3.45E+02	-1.22E+02	2.74E+02	9.91E+02
PW slr indep	2.57E+02	1.82E+02	2.55E+01	2.32E+02	5.88E+02
PW slr normal	2.49E+02	2.12E+02	-1.52E+00	2.04E+02	6.61E+02
PW slr taildep	2.54E+02	2.22E+02	7.50E+00	2.02E+02	6.96E+02

Appendix: some copula details

Tail Dependence

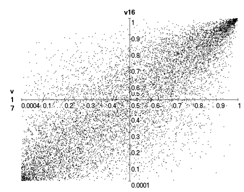
Clayton 0.8

$$C_\alpha(u,v)=\max\left(\left[u^{-\alpha}+v^{-\alpha}-1\right]^{-1/\alpha},0\right)$$

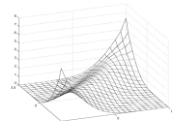


Gumble 0.8

$$C_\alpha(u,v)=\exp\left\{-\left[(-\ln u)^{\alpha}+(-\ln v)^{\alpha}\right]^{1/\alpha}\right\}$$



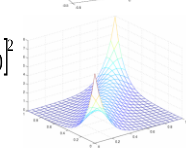
MinInf Copula (ρ=0.8)



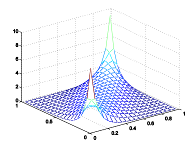
Frank's copula: (ρ=0.8)

$$c(u,v;\theta)=\theta(1-e^{-\theta})e^{-\theta(u+v)}/\left[1-e^{-\theta}-(1-e^{-\theta u})(1-e^{-\theta v})\right]^2$$

$$C(u,v;\theta)=-\frac{1}{\theta}\log\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right)$$



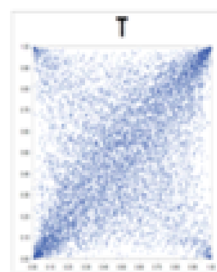
Normal copula: (ρ=0.8)



The T copula

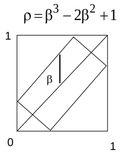
The Student-t copula is an elliptical copula defined as:

$$C_{\rho\nu}=(u,v)=\int_{-\infty}^{t_\nu^{-1}(u)}\int_{-\infty}^{t_\nu^{-1}(v)}\frac{1}{2\pi(1-\rho^2)^{1/2}}\left\{1+\frac{x^2-2\rho xy+y^2}{\nu(1-\rho^2)}\right\}^{-(\nu+2)/2}$$



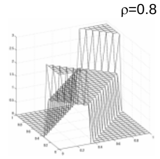
Diagonal Band Copula

$$\rho=\beta^3-2\beta^2+1$$

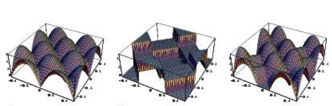


MI(f) = $-\ln(2^{|1-\beta|}|\beta|)$

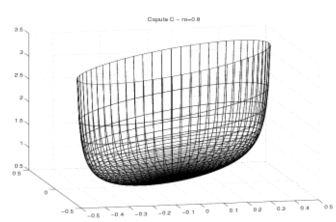
density of diagonal band copula with correlation ρ=0.8



generalized diagonal band (Bajorski; 01)



Elliptical Copula

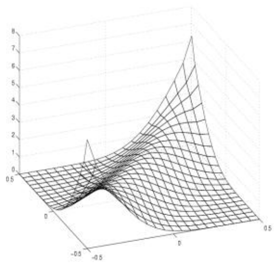


$$f_\rho(u,v)=\begin{cases}\dfrac{1}{\pi\sqrt{\frac{1}{4}(1-\rho^2)-u^2-v^2+2uv}} & (u,v)\in B\\ 0 & (u,v)\notin B\end{cases}$$

$$B=\left\{(u,v)\,|\,u^2+\left(\frac{v-\rho u}{\sqrt{1-\rho^2}}\right)^2<\frac{1}{4}\right\}$$

MI(f) = $1+\ln(2)+\ln(\pi\sqrt{(1-\rho^2)})$

Which copula?



$$\text{maximize} \quad -\sum_{i,j=1}^{n} p_{ij} \ln p_{ij}$$

subject to

$$\sum_{i=1}^{n} p_{ij} = \tfrac{1}{n}, \qquad \sum_{j=1}^{n} p_{ij} = \tfrac{1}{n};$$

$$\sum_{i,j=1}^{n} x_i x_j p_{ij} = t$$

$$(x_i, x_j) \in \left(\tfrac{2i-1-n}{n}, \tfrac{2j-1-n}{n}\right)$$

(M. Fischer, VCH chap 2)

Fully Nested Archemidian Copulae

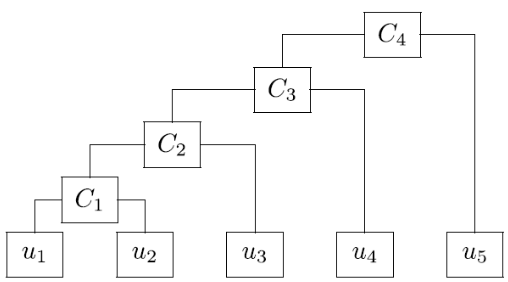


Figure 2.1. FNA copula for $n = 5$.

Simple result on Tail dependence for scale mixures

Let X and Y be positive RVs with survivor function S, and conditionally independent given U ~ unif(0, 1), such that $S(x|u) = S(x)^u$. Then:

$P(X>x \mid Y>x) = (1/2)\,(S(x)^2 - 1) / (S(x) - 1)$.

PF $\forall k>0$:

$\int du\, S(x)^{ku} = S(x)^{ku}/(k \ln(S(x)))\ \Big|_0^1 = S(x)^k - 1 / (k \ln(S(x)))$.

$P(X>x \mid Y>x) = \int du\, S(x)^{2u} / \int du\, S(x)^u$; substitute above expression.

Remark, for any such X,Y, UTD(X,Y) = ½ .

Recall definition: X is subexponential if for $\{X_i\}$ iid copies of X;

$P(\sum_{i=1\ldots n} X_i > x) / P(\vee\{X_1\ldots X_n\} > x) \longrightarrow 1$ as $x \longrightarrow \infty$.

Trick: $P(\vee\{X_1\ldots X_n\} > x) = 1\text{-}F(x)^n = (1\text{-}F(x)) \sum_{k=0\ldots n\text{-}1} F(x)^k \sim nS(x)$ as $x \longrightarrow \infty$.

Proposition. Let $\{X_i, Y_i\}$ be subexponential with survivor function S, and conditionally independent given U ~ unif(0, 1); $S(x|u) = S(x)^u$; let $\sum$ and $\sum$' be sums of n X's and n Y's resp. Then UTD($\sum$, $\sum$') = ½.

PF As above we get $P(\sum > x \mid \sum' > x) = ½ (S_{\sum}(x)^2 – 1) / (S_{\sum}(x) – 1) \sim (x \longrightarrow \infty)$

$½ (n^2S(x)^2 – 1) / (nS(x) – 1) \longrightarrow ½$.

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